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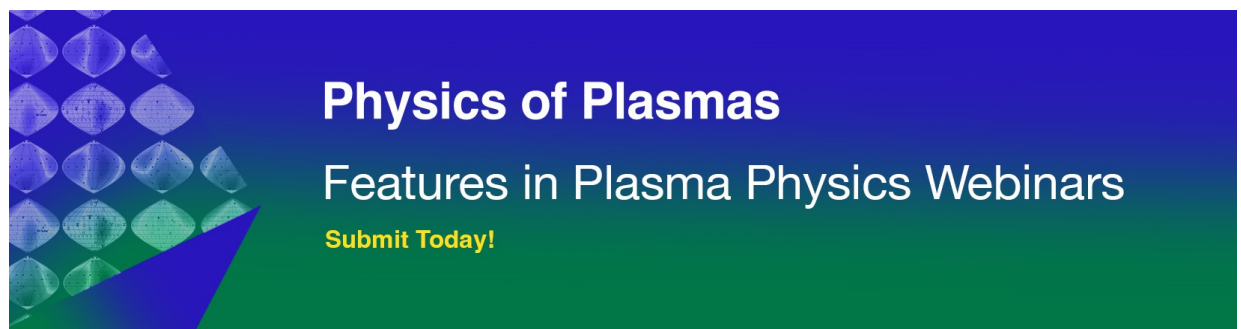
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## ABSTRACT

The two-plate initial boundary-value problem of Parker is reviewed, treating the relaxation of a 3D magnetic field prescribed with an arbitrary topology to a terminal force-free field in a cold, viscous, electrically perfect fluid conductor. Anchored by their foot-points at the perfectly conducting rigid plates, the relaxing field preserves its topology. The Parker magnetostatic theorem states that for most prescribed field topologies, the terminal field must embed current sheets. The elements of this theorem are reviewed and analyzed to relate this initial boundary-value problem to (i) the variational problem for a force-free field of a given topology and (ii) the direct construction of a force-free field in terms of its pair of Euler flux functions. New insights and understanding are presented on the theorem as the compelling basis of the Parker theory of solar coronal heating.

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## I. INTRODUCTION

The fundamental magnetostatic theorem of Eugene N. Parker, hereafter the theorem, states that a 3D magnetic field of complex topology, frozen into an electrically perfect fluid conductor, generally attains force equilibrium by spontaneously embedding infinitesimally thin current sheets (CSs) across which the field is tangentially discontinuous. In the absence of resistivity, the discrete electric current flowing in a magnetic tangential discontinuity (TD), given by Ampère's law, arises naturally. Magnetic flux surfaces move as fluid surfaces. In a 3D evolution to equilibrium, two flux systems separate in space can readily push an intervening third flux system completely out of their way to make direct contact at a TD. The crux of the theorem is that most equilibrium fields must embed TDs by the nonlinear action of the anisotropic Lorentz force under the frozen-in condition. That is, equilibria embedding TDs are the rule rather than the exception. This concept-transforming property was discovered and formulated as a theorem by Parker<sup>1</sup> 50 years ago. The basis for it and its implications were carefully laid out in his monograph<sup>2</sup> with extensive references, hereafter referred to as the monograph.

The theorem originated in the Parker theory of coronal heating under the condition of near-perfect electrical conductivity.<sup>1–6</sup> The adjective near-perfect is taken in the hydromagnetic sense of a large

Lundquist number  $\mathcal{L}_\eta = \frac{U_0 L_0}{\eta}$  in a macroscopic fluid flow of characteristic speed  $U_0$ , length scale  $L_0$  and electrical resistivity  $\eta$ . Treating the corona as fully ionized hydrogen at temperature  $T = 10^6$  K dominated by a  $\sim 10$  G field, the Spitzer resistivity gives  $\eta \sim 0.5 \times 10^{13} T^{-3/2} = 5 \times 10^3 \text{ cm}^2 \text{ s}^{-1}$ . Setting  $U_0 \sim 500 \text{ km s}^{-1}$ , a characteristic coronal Alfvén speed, we see that the corona behaves like a perfect conductor with  $\mathcal{L}_\eta \geq 10^{10}$  down in scale lengths to that of the thermal proton mean-free-path  $L_{mfp} \sim 100$  km. When not disrupted by high-energy flares and Corona Mass Ejections, the low- $\beta$  corona yields to its embedded  $\sim 10$  G field and evolves quasi-statically. The evolution is driven by the heavy photosphere slowly moving at speeds  $\sim 0.5 \text{ km s}^{-1}$  compared to the coronal Alfvén speeds  $\geq 500 \text{ km s}^{-1}$  at which the corona adjusts dynamically to remain in its large-scale near-equilibrium state.

Through the thin ( $10^4 \text{ km} \approx 3 \times 10^{-2} R_\odot$ ) chromosphere, temperature rises steeply from  $\sim 6 \times 10^3$  K at the photosphere to the million degree temperature in the low corona, as the near hydrostatic density falls some 8 orders of magnitude. This gravitationally bound transition layer is rich with time-dependent radiative and hydromagnetic processes nowadays observed with impressive spectral, spatial, and temporal resolutions.<sup>7</sup> We will not treat the chromosphere, interesting in its own right, except to state that, through it, photospheric

magnetic fields extend into the corona, delivering an ever-present Poynting flux at a rate adequate to account for the estimated global loss of  $10^{28}$  erg  $s^{-1}$  in thermal x ray from the corona.<sup>2,3,8</sup> The essence of the heating problem is concerned with how magnetic energy stored in the field under high electrical conductivity might nevertheless be resistively dissipated as a source of heating.

The theorem makes the point that current sheets form irrepressibly in the coronal field under the condition  $\mathcal{L}_\eta \gg 1$ , only to have the condition break down locally wherever a current sheet thins toward zero width. The thinning sheet then dissipates via magnetic reconnection taking place with the otherwise negligible resistivity.<sup>9,10</sup> The dissipation locally heats the corona and, by the removal of the current sheet, the high-conductivity condition  $\mathcal{L}_\eta \gg 1$  is locally restored. Under the theorem, returning the field to the perfect-conductivity regime sets the stage for further formation of current sheets and their subsequent dissipation. These dissipative events explain the ubiquitous, intermittent, small-scale nano-flares observed in the x-ray corona.<sup>2,3</sup>

The present article focuses on the theorem as a basic MHD property while keeping in mind its fundamental role in coronal heating. Section II presents a constructive analysis of the two-plate initial boundary-value problem introduced by Parker,<sup>1</sup> treating the relaxation of a field to a terminal force-free state in a cold, viscous, perfectly conducting fluid.<sup>11</sup> A variational description of the terminal state presents an instructive comparison with the force-free field as an analytical solution to the nonlinear, coupled PDEs describing a global pair of Euler flux functions. The topological properties of the frozen-in fields are central to the theorem. In particular, what is meant by most equilibrium fields will be defined with conceptual precision. Section III presents a concluding discussion, pointing out several basic implications previously not known or appreciated and offering a view that there is interesting work to do on the low corona as a quasi-steadily driven,  $\beta \ll 1$ ,  $\mathcal{L}_\eta \gg 1$  turbulent hydromagnetic atmosphere.

## II. THE PARKER TWO-PLATE PROBLEM

The two-plate problem seeks a time-dependent solution for a field  $\mathbf{B}$  in the unbounded fluid between two parallel, rigid, perfectly conducting Cartesian boundaries at  $z = 0, L$ . The field is unidirectional, streaming from one plate to the other where the magnetic foot-points are rigidly anchored. We treat physically the simplest case of a cold, viscous, perfectly conducting fluid described by

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B} + \nu_1 \nabla^2 \mathbf{v} + \nu_2 \nabla(\nabla \cdot \mathbf{v}), \quad (1)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (2)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}), \quad (3)$$

in standard cgs notations, subject to the boundary-initial conditions

$$z = 0, L, \quad \mathbf{v} = 0, \quad B_z = B_0 > 0, \quad (4)$$

$$R = (x^2 + y^2)^{1/2} \rightarrow \infty, \quad \mathbf{v} \rightarrow 0, \quad \mathbf{B} \rightarrow \mathbf{B}_0 = B_0 \hat{\mathbf{z}}, \quad (5)$$

$$t = 0, \quad \rho = \rho_0, \quad \mathbf{v} = 0, \quad \mathbf{B} = \mathbf{B}_\tau(\mathbf{r}), \quad (6)$$

where  $\mathbf{r} = (x, y, z)$ ;  $\nu_1$  and  $\nu_2$  are constant coefficients of viscosity;  $\rho_0$  is a uniform initial density;  $\mathbf{B}_0$  is a uniform field in the  $z$  direction; and

$\mathbf{B}_\tau$  is an arbitrary prescribed initial field subject to boundary conditions (4) and (5).

The two sketches in Fig. 1 relate the two-plate problem to the magnetic flux tubes populating the quasi-steady solar corona, each tube typically rising from and returning to the photosphere. The initial field  $\mathbf{B}_\tau(\mathbf{r})$  is in a significantly twisted state created from the uniform field  $\mathbf{B}_0$  in a specific manner. Starting with  $\mathbf{B}_0$ , we relax the rigidity of boundary  $z = 0$ . Then, we carry out a continuous, tangential displacement of the magnetic foot-points on  $z = 0$  and spatially continue the displacement into the interior  $0 < z < L$ , while holding  $z = L$  rigid. Taking the foot-point displacement to be incompressible and spatially bounded in the  $x$  and  $y$  directions ensures boundary conditions (4) and (5), respectively. We identify  $\mathbf{B}_\tau$  with the deformed field obtained and restore  $z = 0$  to rigidity.

A noteworthy point to make before proceeding, boundary conditions (4) follows from induction Eq. (3) under the Maxwell requirement for the tangential component of the electric field  $\mathbf{E} = -\frac{1}{c} \mathbf{v} \times \mathbf{B}$  to vanish at  $z = 0, L$  as rigid, perfectly conducting boundaries. Independent of induction Eq. (3), viscosity also demands that  $\mathbf{v} = 0$  at  $z = 0, L$ .

We define  $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$  and obtain the free magnetic energy

$$\mathcal{E} = \frac{1}{8\pi} \int_{0 < z < L} [\mathbf{B} \cdot \mathbf{B} - \mathbf{B}_0 \cdot \mathbf{B}_0] dV = \frac{1}{8\pi} \int_{0 < z < L} \mathbf{b} \cdot \mathbf{b} dV, \quad (7)$$

where  $dV = dx dy dz$ . A product term  $\mathbf{b} \cdot \mathbf{B}_0$  in the integrand has no contribution by virtue of the solenoidal condition  $\nabla \cdot \mathbf{b} = 0$  and boundary conditions (4), the latter implying  $b_z = 0$  at  $z = 0, L$ . Boundary conditions (5) in terms of  $\mathbf{b} \rightarrow 0$  as  $R \rightarrow \infty$  is imposed for a finite free energy  $\mathcal{E}$ . The governing equations (1)–(3) subject to the boundary conditions imply that the sum of the total kinetic energy and free magnetic energy  $\mathcal{E}$  must decrease monotonically by viscous dissipation.<sup>11</sup>

$$\begin{aligned} \frac{d}{dt} \int_{0 < z < L} \left[ \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} + \frac{1}{8\pi} \mathbf{b} \cdot \mathbf{b} \right] dV \\ = - \int_{0 < z < L} [\nu_1 (\nabla \times \mathbf{v})^2 + \nu_2 (\nabla \cdot \mathbf{v})^2] dV. \end{aligned} \quad (8)$$

It follows that as  $t \rightarrow \infty$ ,  $\mathbf{v} \rightarrow 0$ , as the field relaxes to the force-free field  $\mathbf{B}_f(\mathbf{r}) = \lim_{t \rightarrow \infty} \mathbf{B}(\mathbf{r}, t)$  described by

$$(\nabla \times \mathbf{B}_f) \times \mathbf{B}_f = 0, \quad (9)$$

$$\nabla \cdot \mathbf{B}_f = 0. \quad (10)$$

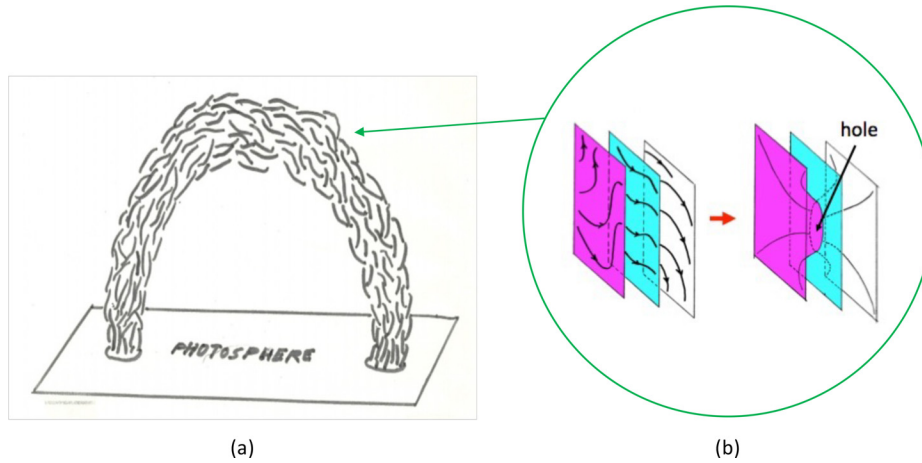
If the terminal field  $\mathbf{B}_f(\mathbf{r})$  is analytic, then the vanishing of the Lorentz force implies

$$\nabla \times \mathbf{B}_f = \alpha \mathbf{B}_f, \quad (11)$$

$$\mathbf{B}_f \cdot \nabla \alpha = 0, \quad (12)$$

respectively, expressing Ampère's law for a field-aligned current density and the solenoidal condition requiring the proportionality function  $\alpha$  to be constant along each field line.<sup>13,14</sup> The function  $\alpha$  defining the current density must (i) yield a field solution  $\mathbf{B}_f(\mathbf{r})$  that meets boundary conditions (4) and (5), and (ii) be demonstrably the result of a continuous deformation of the initial field  $\mathbf{B}_\tau(\mathbf{r})$  by the viscous flow.

The two-plate problem for  $\mathbf{B}_f(\mathbf{r})$  is topological in nature, not a traditional boundary value problem. Let us use the textbook definition



**FIG. 1.** (a) A solar coronal magnetic flux tube with its two feet anchored to the solar photosphere, hand-sketches by E. N. Parker to depict the topologically complex braiding of its internal flux bundles, presented with permission from the Estate of E. N. Parker. The two-plate problem keeps the equilibrium problem geometrically simple by representing such a flux tube as a field located in the unbounded perfectly conducting fluid between rigid planes  $z = 0, L$  where the field is anchored with a uniform distribution of  $B_z$ . (b) A sketch depicting magenta and white, local magnetic flux surfaces pressing into contact through a hole punched into a third, intervening, blue flux surface, discussed in Sec. II C. This subfigure is reproduced with permission from a composite figure in the article<sup>12</sup> B. C. Low, *Sci. China Phys. Mech. Astron.* **58**, 1–20 (2015). Science China Physics, Mechanics & Astronomy.

of topology encompassing the properties of a given geometric object that are invariant under all continuous deformation of the object. Topological properties are formidable to describe explicitly, for example, the mutual linkages among closed tubes of magnetic flux. In the case of the two-plate problem, the mutual field-line mapping of the foot-points fixed on  $z = 0, L$  as well as the interweaving among the anchored flux tubes are invariant under the frozen-in condition. We work around explicitly imposing these invariant properties by the conceptual sufficiency of knowing that the two fields  $\mathbf{B}_\tau(\mathbf{r})$  and  $\mathbf{B}_f(\mathbf{r})$  are topologically equivalent if and only if one is continuously deformable into the other.

A continuum of scalar functions  $\alpha$  exists for which  $\mathbf{B}_\tau(\mathbf{r})$  and  $\mathbf{B}_f(\mathbf{r})$  are topologically equivalent analytical fields. The theorem makes its first essential point here that a topologically equivalent analytical  $\mathbf{B}_f(\mathbf{r})$  generally does not exist for an arbitrary prescribed  $\mathbf{B}_\tau(\mathbf{r})$ . Equation (12) is a hyperbolic PDE for  $\alpha$ , with the field lines as the characteristics such that, in general,  $\mathbf{B}_f(\mathbf{r})$  may not be continuous everywhere; see Chap. III of the monograph.

To account for TDs in the field  $\mathbf{B}_f(\mathbf{r})$ , we formally generalize (11) to the form

$$\nabla \times \mathbf{B}_f = \alpha \mathbf{B}_f + \sum_j \mathcal{C}_j(\mathcal{S}_j), \quad (13)$$

as a statement of Ampère’s law, where  $\mathcal{C}_j$  are a set of discrete current flowing in their respective TDs located on flux surfaces  $\mathcal{S}_j$ . By  $\mathcal{C}_j$  being discrete, it is meant the “profile” of the current density across  $\mathcal{S}_j$  is a Dirac  $\delta$ -function, the current density being unbounded but carrying a finite integrated current.<sup>15,16</sup> This Ampère equation is coupled to Eq. (12) describing the field-aligned current density in the continuous part of the magnetic field.

The discrete currents  $\mathcal{C}_j$  flowing in surfaces  $\mathcal{S}_j$  cannot be arbitrarily defined and inserted into the otherwise continuous force-free field. Decompose the Lorentz force into the sum of magnetic-tension and isotropic-pressure forces,

$$\frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B} = \frac{1}{4\pi} (\mathbf{B} \cdot \nabla) \mathbf{B} - \nabla \left( \frac{\mathbf{B} \cdot \mathbf{B}}{8\pi} \right). \quad (14)$$

The integral of the Lorentz force across a TD surface  $\mathcal{S}_j$  has no contribution from the tension force, with the net force on  $\mathcal{S}_j$  given by the difference between the pressure forces on its two sides. Therefore, the surface  $\mathcal{S}_j$  in force equilibrium requires the magnetic pressure to be continuous across it. That is, the TDs embedded in  $\mathbf{B}_f(\mathbf{r})$  are in equilibrium with the tangential field rotating an abrupt angle at constant amplitude across each TD. This TD equilibrium condition is subject to the requirement that  $\mathbf{B}_f(\mathbf{r})$  given by the Ampère’s law is topologically equivalent to  $\mathbf{B}_\tau(\mathbf{r})$ .

The theorem is stronger than stated above. For most prescribed  $\mathbf{B}_\tau(\mathbf{r})$ , its terminal field  $\mathbf{B}_f(\mathbf{r})$  must embed equilibrium TDs. A variational description of  $\mathbf{B}_f(\mathbf{r})$  offers a useful perspective.

### A. The variational two-plate problem

Consider the deformation of the frozen-in field in terms of the Lagrangian fluid displacements under mass conservation and induction Eqs. (2) and (3). The end result of any continuous deformation of the fluid is a reversible, one-to-one Lagrangian transformation<sup>17</sup>

$$\mathbf{r} = \mathbf{r}(\mathbf{r}'), \quad (15)$$

$$L_{ij}(\mathbf{r}, \mathbf{r}') = \frac{\partial x_i}{\partial x'_j}, \quad (16)$$

$$|L| = \epsilon_{ijk} \frac{\partial x_1}{\partial x'_i} \frac{\partial x_2}{\partial x'_j} \frac{\partial x_3}{\partial x'_k} \neq 0, \quad (17)$$

describing the Cartesian location of each fluid element at  $\mathbf{r} = (x_1, x_2, x_3) \equiv (x, y, z)$  displaced from its previous Cartesian location at  $\mathbf{r}' = (x'_1, x'_2, x'_3) \equiv (x', y', z')$ . Continuity of deformation means that the infinitesimal neighborhood of every fluid element in  $\mathbf{r}'$ -space is transformed onto an infinitesimal neighborhood of the fluid element in

$\mathbf{r}$ -space. Thus, different histories of continuous deformations are topologically equivalent if they produce at the end the same Lagrangian transformation of the fluid elements.

Equations (2) and (3) are equivalent to the Lagrangian displacements of fluid and field

$$\rho(\mathbf{r}) = \frac{\rho'(\mathbf{r}')}{|L|}, \quad (18)$$

$$B_j(\mathbf{r}) = \frac{L_{ij}}{|L|} B'_i(\mathbf{r}'), \quad (19)$$

the prime denoting variables in the Cartesian  $\mathbf{r}'$ -space. The flux surfaces of the field deform as fluid surfaces. Thus, any pair of flux surfaces of  $\mathbf{B}(\mathbf{r})$  transforms into a corresponding pair of flux surfaces of  $\mathbf{B}'(\mathbf{r}')$  sandwiching the same “frozen” fluid between them.

In the two-plate problem, any given continuous field in  $0 < z < L$ , with foot-points fixed at the boundaries, defines a vector-function space  $\mathcal{B}(\tau)$  of topologically equivalent continuous fields, now interpreting  $\tau$  to denote the common topology of the fields. In other words, any two fields in  $\mathcal{B}(\tau)$  are continuously deformable one into the other, related by a reversible Lagrangian transformation (16). Suppose  $\mathcal{B}(\tau)$  contains a unique field with the lowest field energy  $\mathcal{E}$  in  $\mathcal{B}(\tau)$ . Then, this field is the terminal field  $\mathbf{B}_f(\mathbf{r})$  of the time-dependent solution to the two-plate problem. That is, in the space  $\mathcal{B}(\tau)$ , each field  $\mathbf{B}_\tau(\mathbf{r})$  as the initial state is connected by a viscous evolutionary path to the field  $\mathbf{B}_f(\mathbf{r})$ .

We thus arrive at the variational problem,

$$\delta\mathcal{E} \equiv \frac{1}{8\pi} \delta \int_{0 < z < L} \mathbf{b} \cdot \mathbf{b} \, dV = 0, \quad \mathbf{B} = \mathbf{B}_0 + \mathbf{b} \in \mathcal{B}(\tau), \quad (20)$$

that identifies  $\mathbf{B}_f(\mathbf{r})$  as an extremum in  $\mathcal{E}$ . Then, force-free Eqs. (11) and (12) are the extremizing Euler–Lagrangian equations to be solved for  $\mathbf{B}_f(\mathbf{r})$  subject to (i) boundary conditions (4) and (5), and (ii) the condition that  $\mathbf{B}_f(\mathbf{r}) \in \mathcal{B}(\tau)$ ; that is,  $\mathbf{B}_f(\mathbf{r})$  is a continuous field topologically equivalent to a given  $\mathbf{B}_\tau(\mathbf{r})$ .

The theorem states that, in general, the terminal field  $\mathbf{B}_f(\mathbf{r})$  may contain TDs, in which case  $\mathbf{B}_f(\mathbf{r}) \notin \mathcal{B}(\tau)$  and the variational problem (20) defined over the space  $\mathcal{B}(\tau)$  does not have a solution. This is an intrinsic feature of variational calculus.<sup>18</sup> Since  $\mathcal{E}$  is bounded from below, it unquestionably has a minimum value in  $\mathcal{B}(\tau)$ , whereas that minimum may not be realized by any field in  $\mathcal{B}(\tau)$ . In practice, the minimum can be determined by an evolutionary path in  $\mathcal{B}(\tau)$  of a monotonically decreasing  $\mathcal{E}$ , such as defined by a viscosity-driven time-dependent solution to the two-plate problem. If  $\mathbf{B}_f(\mathbf{r})$  contains TDs, the  $\mathcal{E}$ -minimizing path can approach the minimum value as close as desired without leaving  $\mathcal{B}(\tau)$ . Generally, the space  $\mathcal{B}(\tau)$  is mathematically not compact, in the analogous sense of a partially open interval  $0 \leq x < 1$ . In the latter, an infinite series of points can converge as close as desired on the end point  $x=1$  without leaving the interval, despite the fact that this end point does not belong to the interval, whereas an infinite series can converge and arrive at the other end point  $x=0$ , which belongs to the interval.

Specifying the space  $\mathcal{B}(\tau)$  completes the statement of the variational problem (20), without which the extremum condition  $\delta\mathcal{E} = 0$  is neither defined nor meaningful. The analytical force-free field of a topology  $\tau$  satisfying Eqs. (11) and (12) is an instance of the variational problem possessing a solution. However, this field belongs to a

topologically restricted set, for there is the other distinct set of fields in topologies requiring the presence of equilibrium TDs. The existence of the latter set is the reason for the general absence of a solution to the variational problem defined by a specific space  $\mathcal{B}(\tau)$ . The TD-bearing force-free fields are not discontinuous in an arbitrary manner but are identified by their respective spaces  $\mathcal{B}(\tau)$ . They satisfy the Euler–Lagrangian equations as weak solutions and their discontinuities satisfying force balance in the integral sense analogous to the hyperbolic equations of compressible hydrodynamics.<sup>19</sup> Hydrodynamic shocks are weak solutions that satisfy the conservative PDEs in the integral sense of preserving the conserved quantities across the shocks.

The minimum-energy force-free field of any given topology  $\tau$  must physically exist under the frozen-in condition, and this is assured by including the weak solutions to the variational problem (20). Here, we arrive at the practical problem of how a given invariant topology is to be implemented in the direct construction of a force-free field.

### B. Euler flux-function construction of force-free fields

Let us assume analyticity and represent a field by a pair of Euler flux functions,<sup>17</sup>

$$\begin{aligned} \mathbf{B}(\mathbf{x}) &= \nabla U \times \nabla V, \\ &= \frac{\partial(U, V)}{\partial(y, z)} \hat{\mathbf{x}} + \frac{\partial(U, V)}{\partial(z, x)} \hat{\mathbf{y}} + \frac{\partial(U, V)}{\partial(x, y)} \hat{\mathbf{z}}, \end{aligned} \quad (21)$$

automatically satisfying solenoidal condition (10). The representation implies  $\mathbf{B} \cdot \nabla U = \mathbf{B} \cdot \nabla V = 0$ , showing that the level surfaces of  $U$  and  $V$  are magnetic flux surfaces that mutually intersect along field lines. Thus, each field line is a line of constant  $U$  and  $V$ . This geometric relationship also applies to the field-aligned current density of a force-free field, for which  $(\nabla \times \mathbf{B}) \cdot \nabla U = (\nabla \times \mathbf{B}) \cdot \nabla V = 0$ , reducible to

$$\nabla \cdot (\mathbf{B} \times \nabla U) = \nabla \cdot [(\nabla U \times \nabla V) \times \nabla U] = 0, \quad (22)$$

$$\nabla \cdot (\mathbf{B} \times \nabla V) = \nabla \cdot [(\nabla U \times \nabla V) \times \nabla V] = 0. \quad (23)$$

That is, force-free Eq. (9) is equivalent to the two coupled second-order PDEs for  $(U, V)$ ,

$$\nabla \cdot [(\nabla U \cdot \nabla V) \nabla U - |\nabla U|^2 \nabla V] = 0, \quad (24)$$

$$\nabla \cdot [(\nabla U \cdot \nabla V) \nabla V - |\nabla V|^2 \nabla U] = 0, \quad (25)$$

to be solved subject to boundary conditions on  $(U, V)$ .

In terms of  $(U, V)$ , induction Eq. (3) takes the forms

$$\frac{\partial U}{\partial t} + \mathbf{v} \cdot \nabla U = 0, \quad (26)$$

$$\frac{\partial V}{\partial t} + \mathbf{v} \cdot \nabla V = 0, \quad (27)$$

describing each fluid element carrying a pair of unchanging values of  $(U, V)$  in any velocity  $\mathbf{v}$ . Consider the displacement  $T_{i,j}(\mathbf{r}, \mathbf{r}')$  of a fluid element from a previous location  $\mathbf{r}'$  to the present location  $\mathbf{r}$ . Then, induction Eqs. (26) and (27) relate  $\mathbf{B}'(\mathbf{r}') = \nabla' U' \times \nabla' V'$  to  $\mathbf{B}(\mathbf{r}) = \nabla U \times \nabla V$  by

$$[U(\mathbf{r}), V(\mathbf{r})] \equiv [(U'(\mathbf{r}'), V'(\mathbf{r}'))]. \quad (28)$$

Flux surfaces are well defined locally but are not necessarily extendable globally. For example, a 3D field wholly contained in a fluid

domain may have infinitely long, volume filling, ergodic field lines that anywhere do lie on locally well-defined flux surfaces, but these flux surfaces cannot be extended globally. Field representation (21) is convenient if the flux functions ( $U, V$ ) are globally defined, which is the case for the two-plate problem where the initial field  $\mathbf{B}_\tau(\mathbf{r})$  is prepared by a continuous deformation of the uniform field  $\mathbf{B}_0$ .

The flux surfaces of  $\mathbf{B}_0 = \nabla U_0 \times \nabla V_0$  with  $[U_0, V_0] = \sqrt{B_0}[x, y]$  are globally defined, two families of Cartesian planes of constant  $x$  and constant  $y$ . The field  $\mathbf{B}_\tau(\mathbf{r}) = \nabla U_\tau \times \nabla V_\tau$  is prepared from  $\mathbf{B}_0$  by an arbitrary chosen, continuous fluid displacement  $T_{ij}^\tau(\mathbf{r}, \mathbf{r}')$ . The flux surfaces of  $\mathbf{B}_\tau(\mathbf{r})$  are thus also globally defined. The fields topologically equivalent to the prepared  $\mathbf{B}_\tau(\mathbf{r})$  define the space  $\mathcal{B}(\tau)$ . Therefore, the two-plate initial-boundary value problem seeks the terminal force-free field  $\mathbf{B}_f(\mathbf{r}) = \nabla U_f \times \nabla V_f$  as a solution to the force-free Eqs. (24) and (25) for  $[U, V] = [U_f, V_f]$ , subject to boundary conditions

$$z = 0, \quad [U, V] = [U_\tau(x, y, 0), V_\tau(x, y, 0)], \quad (29)$$

$$z = L, \quad [U, V] = [U_\tau(x, y, L), V_\tau(x, y, L)] = \sqrt{B_0}[x, y], \quad (30)$$

$$R = (x^2 + y^2)^{1/2} \rightarrow \infty, \quad [U, V] \rightarrow \sqrt{B_0}[x, y], \quad (31)$$

and subject to the additional requirement that  $\mathbf{B}_f(\mathbf{r})$  and  $\mathbf{B}_\tau(\mathbf{r})$  are topologically equivalent.

There are two Lagrangian transformations in this construction of the force-free field  $\mathbf{B}_f(\mathbf{r})$ , the first being  $T_{ij}^\tau(\mathbf{r}, \mathbf{r}')$  that prepares  $\mathbf{B}_\tau(\mathbf{r})$  from  $\mathbf{B}_0$ . The other transformation denoted by  $T_{ij}^f(\mathbf{r}, \mathbf{r}')$  is defined by the field solution  $\mathbf{B}_f(\mathbf{r})$  as a solution of the above boundary-value problem, which is required to be continuously deformable from the prepared  $\mathbf{B}_\tau(\mathbf{r})$  represented by a relationship

$$[U_f(\mathbf{r}), V_f(\mathbf{r})] \equiv [(U_\tau(\mathbf{r}'), V_\tau(\mathbf{r}'))]. \quad (32)$$

All Lagrangian transformations are reversible, so the reverse of  $T_{ij}^f(\mathbf{r}, \mathbf{r}')$  takes  $\mathbf{B}_f(\mathbf{r})$  into  $\mathbf{B}_\tau(\mathbf{r})$ . The transformation  $T_{ij}^\tau(\mathbf{r}, \mathbf{r}')$  is conceptually relevant to the two-plate problem in ensuring  $\mathbf{B}_\tau(\mathbf{r})$  has globally defined flux functions by virtue of  $\mathbf{B}_0$ , but is indirectly related to  $\mathbf{B}_f(\mathbf{r})$ .

The theorem can be seen by inferring the general absence of an analytical solution  $[U_f(\mathbf{r}), V_f(\mathbf{r})]$  to the above boundary-value problem. Although nonlinear PDEs (24) and (25) are formidable, their basic mathematical features can be understood with the physics of the theorem treated in the monograph and the subsequent works derived from it. It bears reminding that analyticity is an assumption extraneous to the physics of the problem. When the topology of  $\mathbf{B}_\tau(\mathbf{r})$  is such that the  $(U, V)$  boundary value problem yields no analytical solution, the physics of the problem must give up the assumption of analyticity and allow for  $\mathbf{B}_f(\mathbf{r})$  to possess TDs as a weak solution of the force-free equations.

There is an incomplete topological relationship between  $[U_f, V_f]$  and  $[U_\tau, V_\tau]$ , contained in boundary conditions (29) and (30) that demand for  $\mathbf{B}_f(\mathbf{r})$  and  $\mathbf{B}_\tau(\mathbf{r})$  to possess the same mapping of the foot-points on  $z=0$  onto  $z=L$ . Now, the preparation of  $\mathbf{B}_\tau(\mathbf{r})$  from  $\mathbf{B}_0$  endows  $\mathbf{B}_\tau(\mathbf{r})$  with two topological properties, the said foot-point mapping as well as a specific interweaving of the field lines going from boundary to boundary. The mathematical structure of the boundary-value problem does not permit both the foot-point mapping and the field-line interweaving to be imposed on the field solution  $\mathbf{B}_f(\mathbf{r})$ .

Imposing only the foot-point mapping, an analytical solution  $\mathbf{B}_f(\mathbf{r})$ , assuming it exists as determined by the boundary value problem, may or may not be topologically equivalent to the given  $\mathbf{B}_\tau(\mathbf{r})$ . If the two fields are topologically equivalent, we would, of course, have found the proper solution  $\mathbf{B}_f(\mathbf{r})$  for the given  $\mathbf{B}_\tau(\mathbf{r})$ . Their topological equivalence is then described by a specific continuous fluid displacement  $T_{ij}^f(\mathbf{r}, \mathbf{r}')$  that relates their respective pairs of flux functions stipulated by Eq. (32).

There are two other possibilities, one being the case of the existing analytical solution  $\mathbf{B}_f(\mathbf{r})$  found not to be topologically equivalent to  $\mathbf{B}_\tau(\mathbf{r})$ , that is,  $\mathbf{B}_f(\mathbf{r}) \notin \mathcal{B}(\tau)$ . The other case is that the boundary-value problem has no analytical solution. In either cases, the solution to the physical problem must be a field  $\mathbf{B}_f(\mathbf{r})$  embedding equilibrium TDs, whose flux functions  $[U_f(\mathbf{r}), V_f(\mathbf{r})]$  are weak solutions. Then, there is no simple continuous displacement transformation  $T_{ij}^f(\mathbf{r}, \mathbf{r}')$  to speak of. Instead, we must deal with an infinite sequence of fluid displacements  $T_{ij}^{\{\tau \rightarrow f\}}(\mathbf{r}, \mathbf{r}')$  that takes  $[(U_\tau(\mathbf{r}'), V_\tau(\mathbf{r}'))]$  as close to the weak solution  $[U_f(\mathbf{r}), V_f(\mathbf{r})]$  as desired.

### C. A perspective view of TD formation

The frozen-in interaction among three flux systems is basic to TD formation, depicted as a 3D process in Fig. 1(b). The magenta and white flux surfaces belong to two local flux systems that converge to make TD contact through a hole in the green flux surface belonging to a third intervening flux system that has been locally expelled out of the way. Actually, similar holes in a laminated layer of flux surfaces of the third system create a tunnel through which the two converging systems enter to meet each other. A contorted flux surface keeps the three systems topologically invariant and distinct. This and other displaced flux surfaces become a complex of TDs during a dynamical evolution toward a minimum-energy state. Each TD attains force balance by equalizing the magnetic pressures on its two sides as the current density aligns with the continuous part of the field. Here, we present four instructive demonstrations of TD formation to complete our analysis of the two-plate problem.

#### 1. The force-free field as a small-parameter power series

The theorem was discovered in the Parker study of a power-series solution<sup>1</sup> for  $\mathbf{B}_f(\mathbf{r})$  to the two-plate problem

$$\begin{aligned} \mathbf{B}_f &= \mathbf{B}_0 + \mathbf{b} \\ &\equiv B_0 \hat{\mathbf{z}} + \sum_{n=1}^{\infty} \mathbf{b}_n \epsilon^n, \quad \epsilon \ll 1, \\ z &= 0, L, \quad b_{n,z} = 0 \text{ for all } n. \end{aligned} \quad (33)$$

Let us assume analyticity by which  $\nabla \times (\nabla p) \equiv 0$  for any scalar function  $p(\mathbf{r})$ . Therefore, force-free Eq. (14) imposes the necessary condition on the tension force

$$\nabla \times [(\mathbf{B} \cdot \nabla)\mathbf{B}] = 0, \quad (34)$$

for the field to be analytical. In the two-plate problem, the power-series force-free field  $\mathbf{B}_f(\mathbf{r})$  is to be topologically equivalent to the given non-equilibrium field  $\mathbf{B}_\tau(\mathbf{r})$  similarly expressed as a power series. Considering this formidable requirement was superseded by Parker's discovery that the necessary condition (34) by the method of induction implies

$$\frac{\partial}{\partial z} \mathbf{b}_n = 0, \text{ for all } n, \text{ in the limit } \epsilon \rightarrow 0. \quad (35)$$

That is, the force-free field  $\mathbf{B}_f(\mathbf{r})$  can have no  $z$  variation in the limit  $\epsilon \rightarrow 0$ .

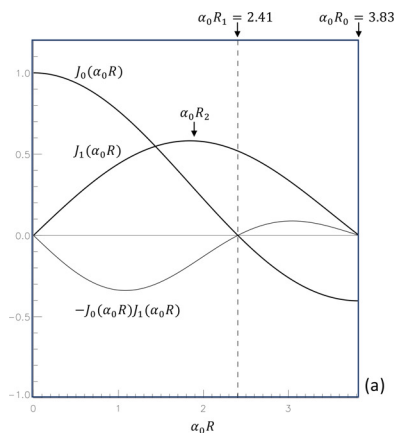
This rigorous result arises physically from the anisotropic nature of the Lorentz force. The tension force is planar at each point in space, lying in the plane containing the tangent and normal to the field line at the point, whereas the pressure force acts volumetrically along the normal to the surface of constant field pressure at the point. Equation (34) is a global condition on the geometric compatibility for the two forces to balance everywhere in space, a compatibility not always possible under the assumption of analyticity.

Let  $\epsilon$  characterize the size of the horizontal spatial variations of the twisted initial field  $\mathbf{B}_\tau(\mathbf{r})$ . Under the assumption of analyticity, the thicknesses of the boundary layers at  $z = 0, L$  may be expected to be of the order of  $\epsilon$  and one might neglect them in the limit  $\epsilon \rightarrow 0$ . Although the result (35) demanding  $\frac{d}{dz} \mathbf{B}_f(\mathbf{r}) = 0$  is obtained in that limit, it reveals that analyticity is generally overly restrictive for a force-free field. The initial field  $\mathbf{B}_\tau(\mathbf{r})$  is arbitrary prescribed, generally possessing an irremovable  $z$ -dependence in its field-line interweaving that is inherited by  $\mathbf{B}_f(\mathbf{r})$ . For field topologies with irremovable  $z$  dependence, their force-free states must embed equilibrium TDs.

Two noteworthy corollaries follow. First, TDs formed in  $\mathbf{B}_f(\mathbf{r})$  must extend along flux surfaces to intersect  $z = 0, L$  where the field intensities may be unbounded, while the TD discrete currents remain bounded, a demonstrable remarkable property of boundary rigidity.<sup>20,21</sup> In other words, the boundary layers at  $z = 0, L$  are mathematically singular in a complex manner. Second, the overly restrictive condition  $\frac{d}{dz} \mathbf{B}_f(\mathbf{r}) = 0$  imposed by analyticity implies that the analytical force-free fields are a continuum that is conceptually of a far smaller size than the continuum of TD-embedding force-free fields.

### 2. Multiple-flux interaction

Consider the Lundquist force-free field,



$$\mathbf{B} = J_1(\alpha_0 R)\hat{\phi} + J_0(\alpha_0 R)\hat{z}, \quad (36)$$

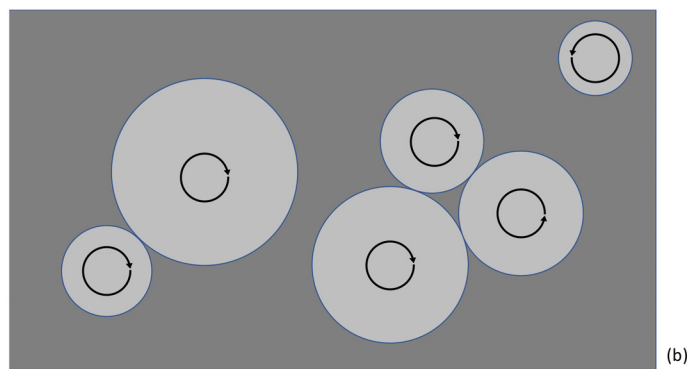
in cylindrical coordinates  $(R, \phi, z)$ , a solution to force-free Eqs. (11) and (12) with  $\alpha = \alpha_0$ , a positive constant. Replacing the  $z$  component with  $B_0$ , a constant, we obtain a non-equilibrium cylindrical field confined within a finite radius

$$\mathbf{B}_{z_0} = J_1(\alpha_0 R)\hat{\phi} + B_0\hat{z}, \quad R \leq R_0, \quad (37)$$

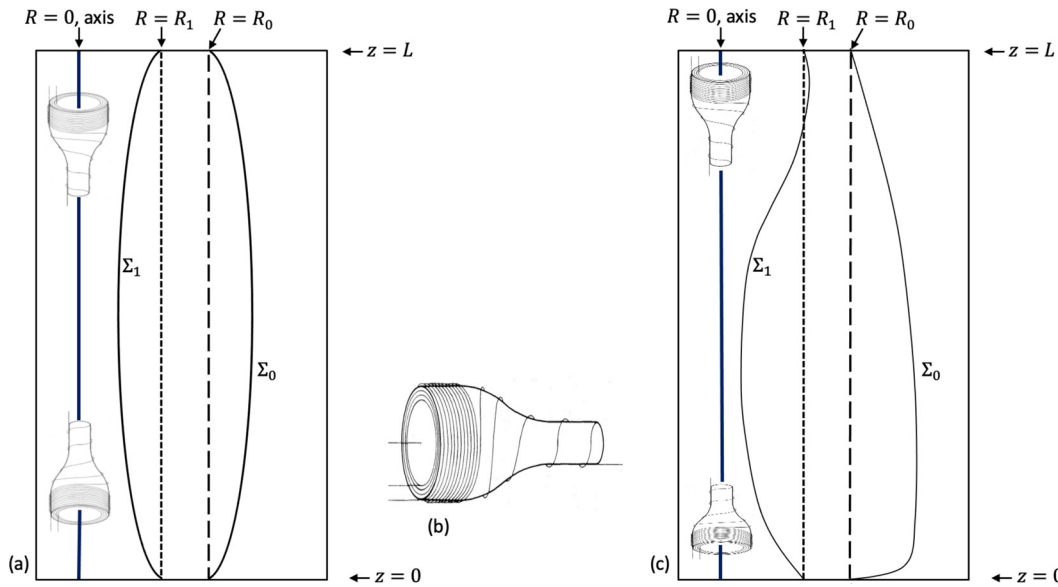
where  $R_0$  is related to the constant  $\alpha_0$  given by the first zero  $\alpha_0 R_0 = 3.83$  of the Bessel function  $J_1(\alpha_0 R_0)$ ; see Fig. 2(a).

A rich variety of the initial, global non-equilibrium fields  $\mathbf{B}_\tau(\mathbf{r})$  for the two-plate problem can be constructed using cylindrical field (37). Each global field varies only with  $(x, y)$  and is composed of the uniform field  $\mathbf{B}_0$  studded with any chosen number of the cylindrical field of an  $\alpha_0$ -value that occupy non-overlapping cylindrical volumes, an example presented in Fig. 2(b). With sufficiently strong twist built into them, the cylindrical fields initially in direct contact along a single field line would expand to create holes in the exterior initially uniform field. Through the holes, different sub-systems of magnetic flux meet at TD surfaces with continuity of equilibrium magnetic pressures across them,<sup>2,22</sup> a canonical case treated on page 162, Fig. 6.26 of the monograph. If  $L$  is sufficiently larger than the cylindrical radii denoted by  $R_0$ , the field located away from the boundaries  $z = 0, L$  would respond to the expansion of the cylindrical fields as described, whereas field displacements are greatly restrained in the two boundary layers along the rigid boundaries. The equilibrium TDs intersect the anchoring boundaries where, under boundary conditions (4), the magnetic pressures on either sides of a TD are integrateably unbounded in a specific form of singularity.<sup>20,21</sup>

Not previously known, TDs can also form inside each cylindrical field, a complicated process possible to infer from the simple case of a single cylindrical field studded into  $\mathbf{B}_0$  depicted in Fig. 3(a). The relaxation of this field proceeds with an initially non-zero Lorentz force in  $R < R_0$ ,



**FIG. 2.** (a) The Lundquist  $B_\phi(R) = J_1(\alpha_0 R)$ . (b) An  $x$ - $y$  sectional view of an initial field  $\mathbf{B}_\tau$  comprising six cylindrical, non-equilibrium,  $z$ -independent fields (light shading) of different cylindrical radii that are studded into the otherwise uniform field (dark shading): three cylindrical fields in touching contacts that isolate an interstitial bundle of uniform external flux, two cylindrical field in single field-line contact, and the sixth cylindrical field studded in isolation. Each cylindrical field is given an arbitrarily assigned handedness in its twist as indicated. In the evolution that follows from initial state (b), all six fields expand outward to seek equilibrium by creating TDs in the manner of flux bundles pushing into contact by punching holes in flux surfaces as depicted in Fig. 1(b) and treated in Sec. II C 3.



**FIG. 3.** (a) The domain  $0 < R, 0 < z < L$  of a single non-equilibrium cylindrical field initially continued at  $R = R_0$  into the external uniform field  $\mathbf{B}_0$  described in text. Relaxation to a force-free state involves the flux surfaces  $\Sigma_0$  and  $\Sigma_1$  deforming, as sketched, from their initial cylindrical shapes (dashed lines) while anchored to  $z = 0, L$  at  $R = R_0$  and  $R = R_1$ , respectively. Symmetry about the mid-plane  $z = 0.5L$  is assumed. (b) A 3D sketch<sup>17,23,24</sup> showing the dynamical accumulation of magnetic twists at the widened end of a cylindrical flux surface that has a gradual spatial increase in its cylindrical radius along the length of the cylinder, reproduced from Parker, *Cosmical Magnetic Fields. Their Origin and Their Activity*. Copyright 1979 Oxford University Press. Reprinted with permission from the Estate of E. N. Parker. This sketch is reproduced in subfigures (a) and (c), appropriately sized and oriented to indicate magnetic twist transport in the pinched field around the axis  $R = 0$ . (c) Consequence of breaking the mid-plane symmetry assumed in the dynamical development in (a). The detail dynamics of the force-free relaxation would determine a critical, intervening flux surface (not shown) between  $\Sigma_0$  and  $\Sigma_1$ , that separates the pinched axial flux from its shell of outer flux, with magnetic twist accumulating toward  $z = 0, L$  in the former, vs accumulating away from  $z = 0, L$  in the latter flux. This critical flux surface as well as  $\Sigma_0$  may be inferred to be among the TDs forming during the relaxation.

$$\begin{aligned}
 (\nabla \times \mathbf{B}_{z_0}) \times \mathbf{B}_{z_0} &= - \left[ \frac{1}{2} \frac{d}{dR} (B_\phi^2) + \frac{B_\phi^2}{R} \right] \hat{R} \\
 &= -J_1(\alpha_0 R) J_0(\alpha_0 R) \hat{R}, \tag{38}
 \end{aligned}$$

as the sum of a pressure force and a tension force defined by  $B_\phi$ . The tension force  $\frac{B_\phi^2}{R}$  acts radially inward everywhere. The pressure  $B_\phi^2$  vanishes at  $R = 0, R_0$  and is maximum at  $R = R_2$ ; see Fig. 2(a), The Lorentz force is thus inward in  $0 < R < R_1$  and outward in  $R_1 < R < R_0, R_1$  defined by the first zero  $\alpha_0 R_1 = 2.41$  of Bessel function  $J_0(\alpha_0 R_1)$  with  $R_2 < R_1$ . The non-equilibrium cylindrical field would thus expand in opposite radial directions from  $R = R_1$ , the radially inward expansion driving a classical pinch and the radially outward expansion pushing into the external, initially uniform field. The rigid anchoring of the field at  $z = 0, L$  immediately results in a significant dependence on  $z$  in the relaxed field. Around the  $R = 0$  axis, the magnetic twist in the narrowed, magnetic flux tube propagates in opposite directions to the boundaries to be accumulated in the funnel-shaped, anchored parts of the flux tube.<sup>17,23,24</sup> In contrast, in the outer cylindrical shell of magnetic flux, away from the  $R = 0$  axis, magnetic twist propagates away from the boundaries to accumulate about the mid-plane  $z = L/2$  where the flux is expanded, sketched in Fig. 3(a).

This intriguing theoretical phenomenon is worthy of further investigation. Figure 3(a) assumes symmetry about the mid-plane. Breaking this symmetry would allow unequal discrete amounts of twist accumulating to the boundaries in the pinched flux around the  $R = 0$  axis, sketched in Fig. 3(c). Similarly, unequal discrete amounts of twist would propagate

from the boundaries to accumulate about the mid-plane where the outer shell of flux has expanded. The critical flux surface separating these two dynamically distinct flux regions would then become a TD.<sup>25</sup>

### 3. The Parker optical analog

Holes in the flux surfaces of a force-free field have an optical analog.<sup>26</sup> The respective flux surfaces of a force-free field  $\mathbf{B}$  and its field-aligned current density coincide. Therefore, the curl of  $\mathbf{B}$  has no component perpendicular to a flux surface, and  $\mathbf{B}$  must be a potential field in the flux surface,

$$\begin{aligned}
 \mathbf{B}_{\text{ff}} &= \nabla \phi(\xi_1, \xi_2), \\
 &= \frac{1}{h_1} \frac{\partial \phi}{\partial \xi_1} \hat{\xi}_1 + \frac{1}{h_2} \frac{\partial \phi}{\partial \xi_2} \hat{\xi}_2, \tag{39}
 \end{aligned}$$

described by orthogonal coordinates  $(\xi_1, \xi_2)$  with the Lamé coefficients  $[h_1(\xi_1, \xi_2), h_2(\xi_1, \xi_2)]$  that define the gradient operator and the path length element  $ds^2 = h_1^2 d\xi_1^2 + h_2^2 d\xi_2^2$ . On this flux surface, the field lines are described by the differential equations

$$\begin{aligned}
 Bh_1 \frac{d\xi_1}{ds} &= \frac{1}{h_1} \frac{\partial \phi}{\partial \xi_1}, \\
 Bh_2 \frac{d\xi_2}{ds} &= \frac{1}{h_2} \frac{\partial \phi}{\partial \xi_2}, \tag{40}
 \end{aligned}$$

where  $B(\xi_1, \xi_2)$  is the field amplitude, whereas  $B^2$  and the potential  $\phi$  are related by the first-order PDE,

$$[\nabla\phi(\xi_1, \xi_2)]^2 = \left(\frac{1}{h_1} \frac{\partial\phi}{\partial\xi_1}\right)^2 + \left(\frac{1}{h_2} \frac{\partial\phi}{\partial\xi_2}\right)^2 = B^2, \quad (41)$$

recognizable as the equation for optical ray paths in a medium with  $\phi$  and  $B$  as the optical eikonal and refractive index, respectively. Thus, differential equations (40) describe the field lines as optical rays streaming and refracted by a variable refractive index  $B(\xi_1, \xi_2)$ .

The bending of light rays around a region of high refractive index (i.e., strong amplitude  $B$ ) has a concavity of the appropriate sense to correspond to the tension force balancing the outward pressure force. This lensing effect can be so severe that the rays are optically excluded from a local region containing a maximum refractive index. In analogy, the balance between the magnetic tension and pressure forces may require the displacement of field lines to open up holes in a flux surface where none existed in the continuous field not in equilibrium. Through those holes, magnetic fluxes external to the flux surface meet one another to form TDs.

Note that TD equilibrium requires  $B^2$  to be continuous; that is, these holes involve no discontinuity in refractive index  $B$  in the optical analog, whereas the vector field  $\mathbf{B}$  is discontinuous across TDs. Not only has analyticity broken down, holes in flux surfaces must appear in the force-free field  $\mathbf{B}_f(\mathbf{r})$  where none existed in the initial field  $\mathbf{B}_\tau$ , two reasons for the  $(U, V)$  boundary-value problem to possess no analytical solution.

#### 4. Multiple-flux interaction in the Cartesian plane

Spontaneous current sheets take a simple form for three or more planar flux systems interacting in the unbounded  $y$ - $z$  space above boundary  $z=0$  where magnetic foot-points are rigidly anchored. The properties discussed below are well known<sup>2,16,20,27</sup> that now take on a new significance when examined in relation to the theorem as developed in Subsections II A and II B.

With  $x$  as an ignorable coordinate, we consider the solenoidal field lying in the  $y$ - $z$  plane with its current density

$$\mathbf{B} = \nabla x \times \nabla A(y, z), \quad (42)$$

$$\nabla \times \mathbf{B} = -\nabla^2 A \hat{\mathbf{x}}, \quad (43)$$

in terms of a flux function  $A(y, z)$ . Field lines are curves of constant- $A$  in the  $y$ - $z$  plane. Independent of any topological consideration, the above current density being everywhere perpendicular to its field

implies that the field, if force-free and analytical, must be potential and unique, satisfying Laplace equation

$$\nabla^2 A = 0, \quad (44)$$

subject to a given distribution of  $A(y, 0)$ , which fixes the normal field component  $B_z$  on  $z=0$ . We also impose the condition that  $\mathbf{B} \cdot \mathbf{B}$  vanishes in the far  $\sqrt{y^2 + z^2} \rightarrow \infty$  to treat fields with finite energy in  $z > 0$ . Let  $\mathbf{B}_{pot} = \hat{\mathbf{x}} \times \nabla A_{pot}(y, z)$  denote the potential field fixed by  $A(y, 0)$ .

Figure 4(a) is a sketch of a three-flux field that is symmetric in  $y$  and not force-free in  $z > 0$ , with a specific  $A(y, 0)$  giving a boundary  $B_z(y, 0)$  that reverses sign on three locations on  $z=0$ . The field has a specific topology  $\tau$  defined by the map of boundary  $z=0$  onto itself from one boundary foot-point to the other of each field line as a curve of constant- $A$ . This field is one of the continuum of continuous fields of all topologies admissible in  $z > 0$  for a fixed  $A(y, 0)$ . In the language of Subsection II A, the vector-function space of all these continuous fields may be decomposed into the union of the disjoint sub-spaces  $\mathcal{B}(\tau)$  each containing fields of the same topology  $\tau$ ,

$$\mathcal{B}_{all} = \bigcup_{\tau} \mathcal{B}(\tau). \quad (45)$$

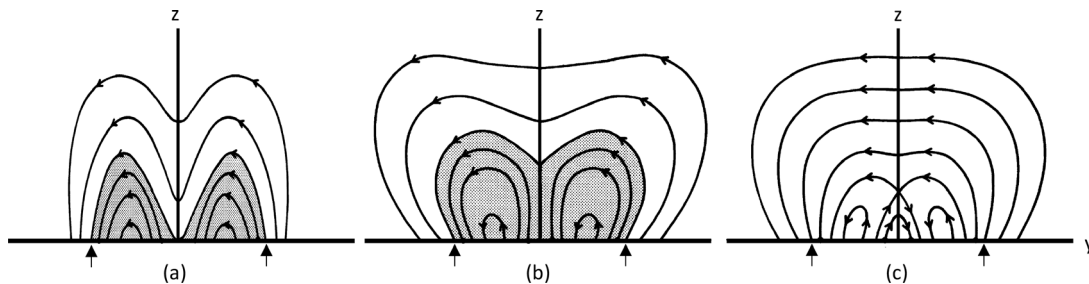
Under the frozen-in condition, with the foot-points anchored rigidly on  $z=0$ , any field is continuously deformable only into another of the same topology  $\tau$ .

The force-free field  $\mathbf{B}_f$  of a chosen  $\tau$  is given by the modified Dirichlet variational principle

$$\delta\mathcal{E} = \delta \int_{z>0} \mathbf{B}_\tau \cdot \mathbf{B}_\tau \, dydz \equiv \delta \int_{z>0} |\nabla A|^2 \, dydz = 0, \quad \mathbf{B}_\tau \in \mathcal{B}(\tau), \quad (46)$$

noting that the variational extremum is taken over the subspace  $\mathcal{B}(\tau)$  instead of the space  $\mathcal{B}_{all}$  of all fields.<sup>16</sup> The given  $A(y, 0)$  alone determines a unique potential field  $\mathbf{B}_{pot}$ , the only analytical force-free field admissible by  $A(y, 0)$ . Therefore, the extremum problem (46) has no analytical solution for  $A$  for all chosen topologies except for the one  $\tau = \tau_{pot}$  of  $\mathbf{B}_{pot}$ . For  $\tau \neq \tau_{pot}$ , the extremum problem has a physical, non-analytical solution  $A$  with discontinuous first derivatives, describing a TD bearing force-free field.

Figure 4(c) is a sketch of the case of a potential field  $\mathbf{B}_{pot}$  determined by  $A(y, 0)$  fixed by the non-force-free field  $\mathbf{B}_{\tau^*}$  in Fig. 4(a). The



**FIG. 4.** Three fields admitted by a fixed boundary  $A(y, 0)$  distribution on  $z=0$ . (a) A non-equilibrium field comprising 3 flux systems with no neutral point. (b) Equilibrium field embedding a vertical equilibrium TD and topologically equivalent to field (a). (c) The potential field comprising 4 flux systems with an X-type neutral point, not topologically equivalent to fields (a) and (b); no other analytical force-free field than field (c) exists for the fixed  $A(y, 0)$ . This figure is reproduced, with a horizontal compression in Fig. 4(a), from Fig. 1 published in the article B. C. Low and R. Wolfson, *Astrophys. J.* **324**, 574 (1988). Copyright 1988 American Astronomical Society.

former is a four-flux field in a topology  $\tau_{pot}$  containing an X-type neutral point, whereas the latter is a three-flux field with a specific topology  $\tau^*$  containing no neutral point, the two fields not topologically equivalent. Figure 4(b) is a sketch of the force-free field  $\mathbf{B}_f$  embedding a TD, given by the weak solution  $A_f$  to the extremum problem (46) for  $\tau = \tau^*$ . Direct calculation shows that  $A_f$  satisfies the Poisson equation

$$\nabla^2 A = f(z)\delta(y), \tag{47}$$

subject to the given  $A(y, 0)$  and the far boundary condition for a finite magnetic energy. Here,  $f(z) \neq 0$  in  $0 < z < z_0$ , describing a vertical current sheet extending to a height  $z_0$ , of a profile such that  $\mathbf{B}_f$  is topologically equivalent to  $\mathbf{B}_{\tau^*}$ . Relating to the two-plate initial boundary-value problem, viscous relaxation would take a field in the initial state  $\mathbf{B}_{\tau^*}$  along an evolutionary path in subspace  $\mathcal{B}(\tau^*)$  to  $\mathbf{B}_f$  as the terminal, TD-embedding force-free field.

This example illustrates the general extreme paucity of analytical force-free states in a perfectly conducting fluid. Fixed by  $A(y, 0)$  describing where a field enters and leaves the domain  $z > 0$ , the fields of all topologies, with just one exception, can attain force-free equilibrium only by embedding TDs. This paucity of analytical force-free states parallels the theorem Parker discovered in his  $\epsilon$ -expansion force-free fields. In the latter, a continuum of analytical force-free fields exists, but they are over restricted to be independent of  $z$ . These fields are sparsely distributed among the continuum of TD bearing  $z$ -dependent force-free fields.

The solenoidal field with  $x$  as an ignorable coordinate may have three components

$$\mathbf{B} = B_x(y, z)\hat{\mathbf{x}} + \nabla x \times \nabla A(y, z), \tag{48}$$

$$\nabla \times \mathbf{B} = -\nabla^2 A \hat{\mathbf{x}} - \nabla x \times \nabla B_x(y, z). \tag{49}$$

Force-free equations (11) and (12) impose the condition that  $B_x(y, z)$  varies as a strict function of  $A(y, z)$  and reduce to the Shafranov equations

$$\begin{aligned} \nabla^2 A + B_x(A) \frac{dB_x(A)}{dA} &= 0, \\ \alpha &= \frac{dB_x(A)}{dA}. \end{aligned} \tag{50}$$

With  $\alpha \neq 0$ , this more general class of force-free fields admits field aligned current densities. In this class, the same basic property of continuous force-free fields being distributed sparsely among the TD force-free fields is found in even richer diversity.<sup>20,21</sup> This general sparsity of continuous equilibrium fields was not recognized at the time the 2D force-free fields were studied to demonstrate that the presence of a magnetic neutral point is not essential for the formation of a TD.<sup>27–29</sup>

### III. OVERVIEW

In the two-plate problem, the perfectly conducting fluid is permanently partitioned into a continuum of distinct sub-volumes bounded by the flux surfaces of the frozen-in field. With the field rigidly anchored at the boundaries  $z = 0, L$ , this permanent magnetic partition of the fluid defines the physical identity of the hydromagnetic system independent of whether it is in static equilibrium or in motion. Therefore, a static equilibrium under the frozen-in condition is not described until the force-balance equation is coupled to the specific

magnetic partition of the fluid of interest. This is the essence of the magnetostatic theorem, concerned with the natural dynamical formation of TDs in a 3D field under the frozen-in condition by the spontaneous expulsion of a flux sub-system out of the way for two other distinct sub-systems to make contact. The generally inevitable TD formation in a field of an arbitrary given topology is the evolutionary means by which a minimum-energy equilibrium is attained. Thus, starting with an analytical field in continuous evolution driven by a monotonically decreasing free magnetic energy, the analyticity of the field and evolution ends with the first formation of a TD whereupon, in the case of the two-plate problem, the analytically defined hydromagnetic equations (1)–(3) must be generalized to account for the existence of the TD having formed as a discrete fluid contact surface.<sup>30</sup> Additional TDs may also form. TD formation being an ideal reversible process in the complete absence of resistive dissipation, an existing TD may be undone dynamically in the course of evolution.

The theorem is concept transforming in its statement that for most field topologies, certain TDs dictated by the specific topology of the field not only must form but must persist to become the equilibrium TDs  $\mathcal{S}_j$  in the generalized force-free Eq. (13). The theorem, beginning with its first mathematical proof for the limited case of  $\epsilon \rightarrow 0$  (Sec. II C 1), is physically established for the general case by the body of works in the monograph. The present article analyzes the theorem seen in the force-free field of a given topology as a solution to a variational problem (Sec. II A) and as a solution to the boundary value problem defined by the force-free field equations (Sec. II B). In both cases, it is inferred that no analytical field solution could be found for most prescribed field topologies, solidly supported by the 2D and 3D demonstrations in Sec. II C. Our analysis gives a new understanding of the theorem applied to the two-plate problem that, in particular, provides a precise physical meaning of the adjective most.

Except for the Gaussian integral and helicity as topological measures<sup>4,12,31–38</sup> of field-line and flux entanglements, it is mathematically intractable to describe the full continuum of the field’s invariant topological properties. In the two-plate problem, the idea of topological equivalent fields provides a means of conceptually defining the unique topology of a field without having to delineate all its topological properties. Consider a similar decomposition (45) of the space of all the continuous fields admissible between the two plates into disjoint subspaces  $\mathcal{B}(\tau)$  of topologically equivalent fields. Then, the topology of any given field is operationally and unambiguously defined by the specific sub-spaces  $\mathcal{B}(\tau)$  to which the given field belongs. Topological equivalence between two fields is explicitly expressed by Eq. (28) in terms of their respective Euler flux functions. The parameter  $\tau$  that labels the subspaces  $\mathcal{B}(\tau)$  is thus also a label for the topology common to the topologically equivalent fields of each subspace. The mathematical language is now in place to meaningfully speak of  $\mathcal{B}(\tau)$  containing the deformed states of a field identified by its invariant topology  $\tau$ .

The continuum of analytical force-free fields being sparsely immersed into the “even larger” continuum of TD-embedded force-free fields has a probabilistic interpretation.<sup>6,25</sup> A field randomly picked from the continuum of all the admissible fields in the hydromagnetic system has essentially zero chance of being in a topology compatible with an analytical force-free field. This corollary is the basis for the assertion that, following the dissipation of a TD under a breakdown of the condition  $\mathcal{L}_\eta \gg 1$ , the likelihood returns for TDs to again form spontaneously under the restored high-conductivity condition.

This is the attractive feature of the Parker theory of coronal heating as a turbulently sustained hydromagnetic process. By this process, near perfect conductivity is as much a consequence of as it is the origin of the maintained high temperature of the fully ionized corona.

The theorem is closely related to the behaviors of a slowly driven, low- $\beta$ , high Lundquist number  $\mathcal{L}_\eta$ , resistive MHD turbulence discovered in high-resolution numerical studies employing the reduced-MHD approximation.<sup>39–41</sup> In context of the two-plate problem, we are looking at a turbulence driven by the plates that are no longer rigid, still heavy and perfectly conducting, and executing large-scale, random, horizontal displacements of magnetic foot-points at well below the characteristic Alfvén speeds in  $0 < z < L$ . If TDs do not form in the 3D turbulent interior field, the foot-point motions can unlimitedly increase the magnetic energy trapped in the large-scale part of the turbulent field. The theorem argues against this behavior on two counts. Not only must TDs form readily under condition  $\mathcal{L}_\eta \gg 1$  to set the stage for resistive magnetic reconnection and dissipative heating, the reconnected field in its changed topology would remain chaotic with its propensity to form fresh TDs. This propensity is separate from the rates at which reconnection<sup>9,10</sup> can proceed as a function of the Lundquist number. If the reconnection is capable of removing TDs sufficiently rapidly, there is then the possibility of a fully developed interior turbulence whose magnetic energy is statistically steady by a fluctuating balance between foot-point energy input and magnetic energy dissipation, the latter controlled by the physics of TD formation and resistive reconnection. This possibility was discovered independently in simulations<sup>39</sup> showing rates of energy dissipation increasing as a moderate fractional power of a relatively low Lundquist number. Later simulations<sup>40,41</sup> capable of achieving significantly higher Lundquist numbers discovered parametric regimes where the dissipation rate is weakly sensitive to the Lundquist number. Although TD formation and resistive reconnection cannot be directly simulated, these simulations may be assessed to be sufficient for discovering the possibility of a self-maintained, resistively heated MHD turbulence, in corroborative agreement with the Parker theory of coronal heating.

Irrepressible spontaneous current sheets also occur in more complex hydromagnetic systems, for example, in the presence of gravity, fluid pressure, steady fluid flows, and complex thermal-energy transport.<sup>42,43</sup> As Parker<sup>2</sup> emphasized in the preface to his monograph, while studies of analytical fields are interesting in their own rights, fields forming TDs that dissipate by an otherwise insignificant resistivity are the rule rather than the exception under astrophysical circumstances.<sup>44–50</sup> To solve the mathematical problems surveyed in Sec. II beyond the analytical success we happen to have, direct numerical simulation holds the only realistic promise. The physical understanding of the theorem provides guidance in formulating numerical simulations<sup>11,51–54</sup> capable of handling the required, orders of magnitudes range in computational Lundquist number  $\mathcal{L}_\eta$ .

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## AUTHOR DECLARATIONS

### Conflict of Interest

The authors have no conflicts to disclose.

### Author Contributions

**Boon Chye Low:** Conceptualization (lead); Investigation (lead); Methodology (lead); Writing – original draft (lead); Writing – review & editing (lead).

## DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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