

BC Low's Unedited Conversations with Eugene N. Parker¹ #1 - #14
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Conversations with Eugene N. Parker¹

I. The Parker 1972 ApJ Paper on Topological Dissipation

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The magnetostatic theorem is described in broad physical terms that a magnetic field in a complex twisted topology is generally without equilibrium in a perfectly conducting fluid; Topological dissipation and the small-scale fields in turbulent gases, E.N. Parker ApJ 174, 499-510. A pedagogic understanding of the theorem is pursued with particular attention to its conceptual content. This is revised from the March 28, 2020, the only revision being the sentence "Reconnection is a chaotic turbulent process" in item No. 6 on Many Other Issues.

1. Statement of the problem. Consider a uniform field $\mathbf{B}_0 = B_0 \hat{\mathbf{z}}$ in a perfectly conducting fluid between two infinite, parallel, perfectly conducting, rigid plates $z = \pm L$. We seek the equilibrium state for a field \mathbf{B}_{init} created by first relaxing the rigidity one of the two plates, say, $z = -L$, where the magnetic foot points are *continuously* displaced arbitrarily under incompressibility, preserving the boundary condition $B_{\text{init},z} = B_0$. Denote by \mathbf{u} a continuous velocity in $-L \leq z < L$ over a period of time $0 < t < t_0$ taking the field from \mathbf{B}_0 to the state \mathbf{B}_{init} governed by the ideal induction equation. The prescribed foot-point velocity $\mathbf{u}_{\text{fp}} = \mathbf{u}|_{z=-L}$, strictly tangential on $z = -L$, winds up and braids the flux tubes of \mathbf{B}_0 . The field having reached the state \mathbf{B}_{init} at $t = t_0$, the boundary $z = -L$ is then restored to rigidity and, by the usual language of the frozen-in condition, any subsequent continuous deformation of the field from the state \mathbf{B}_{init} preserves the field topology denoted by T built into \mathbf{B}_{init} . We will shortly clarify what is meant by field topology.

The Parker Magnetostatic Problem demands for a continuous deformation \mathbf{v} in $-L < z < L$ under the induction equation, subject to rigid boundary condition $\mathbf{v}|_{\pm L} = 0$, required by Maxwell's equations, that kinematically takes the field from the initial state \mathbf{B}_{init} to a terminal equilibrium state with the preserved topology T .

2. Force-free fields. For our purpose here, we take the fluid to be inertia-less and the terminal equilibrium state is a force free field $\mathbf{B} = \mathbf{B}_{\text{ff}}$ satisfying

$$(\nabla \times \mathbf{B}_{\text{ff}}) \times \mathbf{B}_{\text{ff}} = 0, \quad (1)$$

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subject to $B_{\text{ff},z} = B_0$ at $z = \pm L$. The Parker problem thus seeks a solution of equation (??) subject to the boundary conditions such that it has the preserved topology T created in \mathbf{B}_{init} .

A fundamental property of the perfectly conducting fluid should be pointed out. Tangential discontinuities (TDs) in \mathbf{B} are physically admissible, a TD being a fluid surface Σ , also a magnetic flux surface, across which \mathbf{B} is tangentially discontinuous. Ampere’s law then implies the presence of a current sheet (CS), a finite discrete current, flowing along within the TD that can persist with no resistive dissipation. Integrating the Lorentz force across the TD gives the equilibrium condition that the magnetic pressure B^2 must be continuous across the TD for that fluid surface Σ to be force-free everywhere on it. The two tangential vector fields on the opposite sides of Σ have equal amplitudes but unequal orientations. So, in general, the force-free equation (??) must be supplemented by the TD equation

$$\Delta B^2|_{\Sigma} = 0, \tag{2}$$

determining the shape of Σ , if any TD is present, a non-local problem relating Σ to the global field solution. The proper global field is given by Ampere’s law in terms of the continuous part of the force-free current density and the discrete TD currents. The Parker problem is thus faced with the issue of whether the force-free field it seeks to possess the specific topology T of the given \mathbf{B}_{init} can be found in the continuous solutions of the force-free equation (??) without the presence of TDs. In other words, the issue is whether there can be topologies T in the given continuous \mathbf{B}_{init} such that the field deformed from \mathbf{B}_{init} cannot find a force-free state except by the spontaneous formation of TDs. This fundamental issue is addressed by the Parker Magnetostatic Theorem.

3. Field topology. We adopt the textbook definition of topology to comprise all properties of a geometric object that are invariant under all continuous deformation of the object. Thus, e.g., a pair of toroidal tubes of fluid linked N -integer times remain so linked under all continuous deformations. The frozen-in condition of perfect conductivity means that, with the foot-points of \mathbf{B}_{init} rigidly fixed at $z = \pm L$, the fluid is permanently partitioned into the continuum of magnetic flux tubes anchored at the boundaries, each flux tube and each of its sub-tubes all permanently identifiable by the fluid contained therein, respectively. Thus, when the field is deformed continuously under the induction equation, all topological properties of the (complex) braidings among identifiable flux tubes of fluid are unchanging. It is mathematically formidable to explicitly identify all the topological invariants, which must be described by integrals along the paths of flux tubes and field lines, and there is a continuum of such invariant integrals. Even without the complication of TDs, to construct an everywhere continuous force-free field requires solving the force-free equation (??) coupled to the integral equations expressing T , a boundary value problem distinct from the traditional

ones posed by pure PDEs.

Despite the intractability of expressing T by integral equations, the invariance of T is conceptually well defined in the field evolution governed by the induction equation under the action of a continuous velocity. Any two fields realized in $-L < z < L$ with foot-points rigidly anchored at the boundaries are topologically identical if one can be continuously deformed into the other by some suitable continuous velocity. In other words, we have an operational definition of T , strategically avoiding a mathematical expression of it.

We arrive at a book-keeping classification of the fields \mathbf{B}_{init} of topology T created by all the prescribed foot-point displacements $\mathbf{u}_{\text{fp}} = \mathbf{u}|_{z=-L}$ during the preparation period $0 < t < t_0$ when the rigidity of the boundary $z = -L$ is relaxed and then restored. For any \mathbf{B}_{init} so created, by further deforming the field from \mathbf{B}_{init} with any continuous velocity \mathbf{v} meeting boundary conditions $\mathbf{v}|_{z=\pm L} = 0$ for the fixed foot-points, we generate a function space $\mathcal{S}(T)$ of all the continuous fields \mathbf{B} in the fluid with the same boundary flux distribution $B_z|_{z=\pm L} = B_0$ and sharing the common topology T . Denote by \mathcal{S}_{all} the vector function space of all the continuous fields in the fluid of all topologies with the same boundary flux distribution $B_z|_{z=\pm L} = B_0$, i.e., $\mathcal{S}_{\text{all}} = \bigcup_T \mathcal{S}(T)$.

4. The Parker Magnetostatic Theorem states that for most cases of topology T created in \mathbf{B}_{init} , an everywhere continuous force-free field \mathbf{B}_{ff} cannot be found in $\mathcal{S}(T)$. In such a case, the force-free state available to \mathbf{B}_{ff} must contain one or more TDs, not any discontinuity in the global field but TDs subject to the free-surface condition (??) such that the continuous force-free current density and the discrete TD currents give a global field in topology T . This terminal force-free field with TDs *is discontinuous* and does not belong to the vector-function space $\mathcal{S}(T)$ of *continuous* fields. The discontinuous terminal state is the limit approached by the field continuous deforming in $\mathcal{S}(T)$ constrained to remove the Lorentz force everywhere. The fundamental point is that the set of all the topologies \mathcal{T}_{all} created in \mathbf{B}_{init} is composed of two disjoint subsets, i.e., $\mathcal{T}_{\text{all}} = \mathcal{T}_{\text{ff}} \cup \mathcal{T}_{\text{td}}$, the self-explanatory subscripts indicating, respectively, the topologies realizable in everywhere continuous force-free fields and the other topologies realizable only in force-free fields with TDs.

A mathematical aside, the function space $\mathcal{S}(T)$ is like an open unit interval $0 < x < 1$ on the real line, not Cauchy complete in the sense of an infinite sequence of real numbers $x_n, n = 1, 2, 3, \dots$ in the interval converging on $x_\infty = 1$ has a limit not belonging to the interval. A then young Princeton mathematician Chen Xiu Xiong once told me that Cauchy complete function spaces are pretty much understood and the Cauchy non-complete function spaces are where life is exciting.

The PDE-integral boundary value problem for a force-free field gives a continuous force-

free field if $T \in \mathcal{T}_{\text{ff}}$ and a weak solution for a force-free field embedding TDs if $T \in \mathcal{T}_{\text{td}}$. The term weak solution is mathematically meant that force-free equation (??) is, in fact, also satisfied, but in the integrated sense for the TDs. The hydrodynamic shock is a well known example of a weak solution. The TDs are contact fluid surfaces separating distinct subsystems of magnetic flux, the discrete currents in these TDs existing because resistivity is zero.

5. The Parker Proof of the Theorem proceeds by first assuming that a continuous analytic solution exists for a force-free field of topology T in the ϵ infinitesimal neighborhood of the uniform field \mathbf{B}_0 in the vector function space $\mathcal{S}(T)$, represented by

$$\mathbf{B} = \mathbf{B}_0 + \sum_{n=1}^{n=\infty} \epsilon^n \mathbf{b}_n, \quad \nabla \cdot \mathbf{b}_n = 0 \text{ for all } n. \quad (3)$$

For our purpose here, we take pressure to be zero; the Parker proof includes the presence of a magnetostatic pressure. The proof identifies ϵ with the assumed small sizes, compared with L , of the foot-point displacements at $z = -L$ that create the topology T , but the proof can be taken with no loss of generality if we interpret ϵ to be just a small parameter characterizing the infinitesimal neighborhood about \mathbf{B}_0 in $\mathcal{S}(T)$. In the limit of an extremely large L , we follow the proof in further assuming that we may neglect the boundary layers at $z = \pm L$ that analytically accommodates the transition from the interior field to the boundary anchoring of the field, i.e., the boundary layer is thin but smoothly varying with no singularities. The proof then shows that the force-free state imposes upon expansion (??) the necessary condition that the interior field, if force-free, must be invariant in the z direction to all orders of ϵ in the limit of $L \rightarrow \infty$. The history of braiding the flux tubes of the uniform field \mathbf{B}_0 is completely arbitrary and it is conceivably not forbidden to have the braidings varying with z along the flux tubes.

Hence, we have a contradiction between the mathematics and the physics. The resolution is the rejection of assuming the field can be analytic for all topologies T when kinematically deformed to achieve equilibrium under the frozen-in condition. This contradiction, deriving from a *necessary* condition for the force-free state, has been shown to all orders in ϵ , conveniently bypassing the formidable mathematics of expressing T explicitly. To be sure, there is a continuum of force-free fields everywhere continuous in the ϵ -neighborhood but these equilibrium fields must be invariant in z , revealed in the limit of $L \rightarrow \infty$. Thus, the theorem concludes that fields possessing an even greater set of field topologies must contain inevitable TDs in their equilibrium states, namely, those topologies not compatible with invariance in z .

The monograph "Spontaneous Current Sheets in Magnetic Fields, E. N. Parker (1994), Oxford University Press, contains many illustrative examples of how multiple flux-tube in-

teractions during a field evolution may readily result in two flux tubes first forcing a third flux tube out of the way from between them and then coming into contact over a flux surface. The fields on the two sides of the contact surface are generally oriented in different directions to preserve the frozen-in topologies of the their respective flux systems, whereas force-balance of the resultant TD requires the field amplitudes being equal on the two side of the TD. The two subsets \mathcal{T}_{ff} and \mathcal{T}_{td} are infinite. It is intuitively clear in the Parker proof that z -invariance is so restrictive that \mathcal{T}_{ff} is a subset of \mathcal{T}_{all} of measure zero.

6. Many other issues are important and interesting but the Parker proof has constructed a self-contained, physically complete picture. In astrophysical fields in which resistivity is not zero but is small in the sense of $\mathcal{R}_m \gg 1$, the topological formation of TDs is found intermittently wherever and whenever the perfectly conducting fluid approximation applies, whereupon resistivity sets in at dynamical timescales when field gradients are sufficiently intense; such TD formations never reach the mathematical singularity of a discontinuous \mathbf{B} . Magnetic reconnection then dissipates magnetic free energy to heat the plasma as the field topology T changes. Here the implication that \mathcal{T}_{ff} is a subset of \mathcal{T}_{all} of measure zero is far reaching. Reconnection is a chaotic turbulent process, a reconnected field being as unlikely to have a field topology belonging to \mathcal{T}_{ff} as the TD-forming field that produced the reconnection in the first place. In other words, once initiated, reconnection becomes a turbulently never ending process unable to deliver T from \mathcal{T}_{td} into \mathcal{T}_{ff} . So, the name topological dissipation is truly appropriate. If there is only a finite amount of free magnetic energy available under various macroscopic constraints or conservation laws, topological dissipation of magnetic energy is not suppressed, the TDs are inevitable by their topological nature, but they form in intensities depending on the availability of free magnetic energy.

7. The Global Corona The understanding of topological dissipation coupled to the energy constraint on TD formation all at once opens up the physics of magnetic energy storage in the corona and the rich observations of coronal heating, solar flares, CMEs. The changing coronal field reverses its global polarity in concert with the eleven-year sunspot cycles. That the corona, so highly conducting can achieve this reversal in concert and also can also resistively heat itself to its million degree temperatures makes this area of astrophysics truly fruitful and exciting. This concluding statement is the theme of my Chapter 6, titled Coronal Magnetism as a Universal Phenomenon, for the book on the Sun as a Guide to Stellar Physics, published last year: http://download.hao.ucar.edu/pub/low/Chapter6/TheSun_Chapter6_BCLow.pdf.

Conversations with Eugene N. Parker¹

II. Parker Magnetostatic Theorem - Variational Formulation

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The Magnetostatic Theorem of Parker (*Topological dissipation and the small-scale fields in turbulent gases*, 1972, ApJ 174, 499-510) is first expressed in broad terms as a variational problem followed by a historical interlude into the Dirichlet Principle for a precise meaning of a variational extremum. The physical and topological elements of the Theorem are all in place in the exact 2D Cartesian case (*Electric current sheet formation in a magnetic field induced by continuous magnetic footpoint displacements*, Low 1986, ApJ 323, 358; *Spontaneous formation of electric current sheets by the expulsion of magnetic flux*, Low 1989, ApJ 340, 558), capturing the general inevitability of TD formation via multiple-flux interactions in topologically complex 3D fields treated in the 1972 Parker Theorem and the book *Spontaneous Current Sheets in Magnetic Fields*, E. N. Parker, 1994, Oxford University Press.

1. The Parker Variational Problem. Consider a magnetic field \mathbf{B} in a simply connected domain V containing a perfect fluid conductor with a perfectly-conducting rigid boundary ∂V . Let the field lines of \mathbf{B} be all anchored each at a pair of foot-points on ∂V . The field may be deformed by any continuous fluid velocity \mathbf{v} according to the induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}), \quad (1)$$

subject to 2 rigid-boundary conditions that, on ∂V , $\mathbf{v} = 0$ and the normal flux distribution $\mathbf{B} \cdot \hat{\mathbf{n}} = B_n(\partial V)$ is fixed for all time, $\hat{\mathbf{n}}$ being the boundary unit normal.

Essential is the property that any field realized in V for a given $B_n(\partial V)$ cannot change its topology T which we define with some conceptual care as follows. Under induction equation (1), magnetic flux is frozen into the fluid. Thus, every magnetic flux tube has a physical identity by the fluid permanently contained in it and by the fluid permanently contained in each of the sub-tubes of any tube. Topology T comprises all those properties of a geometric object that are invariant under all continuous deformation of the object.

As an example of these invariant properties, induction equation (1) under the rigid-boundary conditions preserves the mapping $(\xi_2, \zeta_2) = [F(\xi_1, \zeta_1), G(\xi_1, \zeta_1)]$ of ∂V onto itself,

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where (ξ_1, ζ_1) and (ξ_2, ζ_2) denote the pairs of boundary foot-points of the field lines. The mapping functions $(F, G) : \partial V \rightarrow \partial V$ are fixed for any specific field \mathbf{B} evolving in V . To express this topological invariance explicitly, we may use Cartesian coordinates $\mathbf{X} \equiv (x, y, z)$ in terms of which $[\mathbf{X}_1(\xi_1, \zeta_1), \mathbf{X}_2(\xi_2, \zeta_2)] \equiv [(x_1, y_1, z_1), (x_2, y_2, z_2)]$ are the coordinates of the respective pairs of foot-points $[(\xi_1, \zeta_1), (\xi_2, \zeta_2)]$. Then, integrating along each field line, we obtain

$$\mathbf{X}_2 - \mathbf{X}_1 = \int_0^{s_2} \frac{\mathbf{B}}{B} ds, \quad (2)$$

where s denotes the path length along the field line, $s = 0$ at foot-point (ξ_1, ζ_1) and $s = s_2$, the full length of the field line at foot-point (ξ_2, ζ_2) . Integral (2) relates the fixed mapping functions (F, G) to a particular invariant attribute of \mathbf{B} . In other words, (F, G) is a permanent attribute of the evolving \mathbf{B} , only one in a continuum of topological attributes, denoted by T , that completely identify the field physically. The other attributes are formidable integrals describing the braidings among any two or more flux tubes that are invariant during the continuous deformation \mathbf{B} . Two fields that differ by any one or more of these topological invariants are not topologically equivalent, meaning that one field cannot be continuously deformed into the other in V . Only fields of the same T are continuously deformable one into another in V .

Topology T is thus conceptually specific without a need to derive the topological invariants explicitly. It also means that the space $\mathcal{S}_{\text{all}} [B_n(\partial V)]$ of continuous solenoidal fields defined by their common boundary-flux distribution $B_n(\partial V)$ is a union of a continuum of disjoint subspaces $\mathcal{S} [B_n(\partial V), T]$ of fields of the same topology T in each subspace, that is, $\mathcal{S}_{\text{all}} = \bigcup_T \mathcal{S}(T)$, keeping the notation uncluttered by not explicitly expressing the dependency on $B_n(\partial V)$. The fields of different T in V are distinct physical entities under the condition of perfect conductivity.

Identifying a specific field in V by its topology T and boundary-flux $B_n(\partial V)$, the Parker Variational Problem seeks to determine its equilibrium state. Taking the fluid to be inertialess, equilibrium is defined to be an extremum in the magnetic energy

$$\delta \int_V B^2 dV = 0 \quad (3)$$

in subspace $\mathcal{S}(T)$. If we may assume that an extremum solution \mathbf{B}_{ff} exists as an analytical field in $\mathcal{S}(T)$, then the usual analytical derivation for the Euler-Lagrangian equation for the extremum gives the force-free equation

$$(\nabla \times \mathbf{B}_{\text{ff}}) \times \mathbf{B}_{\text{ff}} = 0 ; \quad \nabla \cdot \mathbf{B}_{\text{ff}} = 0, \quad (4)$$

subject to the prescribed boundary-flux $B_n(\partial V)$ and the specific topology T that identifies $\mathcal{S}(T)$ to be the vector-function space over which the magnetic energy is an extremum.

The following technical point is worth pointing out. The continuum of integral equations expressing the invariants associated with T may be explicitly coupled to the variation (3) via the use of Lagrangian multipliers. Again assuming that \mathbf{B}_{ff} is analytic, the solutions to the force-free equation (4) have to be searched among them for the one or more solutions possessing *all* the invariants of T specific to the field. Although there may be particular (analytical) field solutions in this construction, the intuitive expectation seems clear that the governing system of PDE-integral equations is, in general, over determining without an analytical solution in the traditional sense of mathematical analysis for every instance of T . This expectation is solidly valid, the mathematical essence of the Parker Magnetostatic Problem.

In fact, this non-existence of the analytical solution is basic to the the physics of ideal hydromagnetics on two counts. That there should be a continuum of topological invariants under perfect conduction is best appreciated by reminding ourselves that magnetic flux conservation is the fundamental property of induction equation (1). That is, every deforming fluid surface Σ conserves the magnetic flux across it. Therefore, there is a continuum of conserved quantities or invariants in a magnetic field in the perfectly conducting fluid. The use of magnetic helicity to describe the topology of a field is powerful but also encumbering when we forget that flux conservation is the physical property on which all ideas of field topology are based; see *Field topologies in ideal and near ideal magnetohydrodynamics and vortex dynamics*, Low, Sci. China, Phys. Mech, Astron. 58, 5626, 2015. The other fundamental point is that TDs and CSs are physically and mathematically admissible in the perfectly conducting fluid, with resistivity rigorously zero. From the physics point of view, the variational formulation based on continuous fields actually goes far in capturing the basic physical principles, but the inevitable TDs are physically a crucial step to complete the physics being described.

If the variation (3) is taken over all the continuous fields of \mathcal{S}_{all} , irrespective of field topology T , there is then no issue of over determination and the solution is the unique potential field $\mathbf{B}_{\text{pot}} = \nabla\Phi$ satisfying the Laplace equation

$$\nabla^2\Phi = 0 \tag{5}$$

subject to the boundary condition $\nabla\Phi \cdot \hat{\mathbf{n}} = B_n(\partial V)$. Moreover, the extremum is a minimum, that the potential field has the lowest energy of all the fields in V sharing the same boundary-flux. Although there is superficially no issue of an over-determined problem, there is still ground for worrying whether TDs are inevitably present. One is accustomed to think of a potential field as a vacuum field, but a potential field is as much a field in a perfectly conducting fluid in which electric current happens to be absent. This was a big deal in modern analysis, separating the existence of a value of an extremum from whether that

value can be realized in the function space of the variation. We take an interlude on this milestones in modern analysis which is insightful for our understanding the Parker Theorem.

2. The Dirichlet Principle. In his 1849 doctoral thesis that laid the foundation for the theory of a function of a complex variable, Riemann used the Dirichlet Principle in a simple proof that a solution always exists to the above potential boundary value problem. The Principle was introduced by Dirichlet in a Goettingen lecture attended by Riemann. The variation

$$\delta E \equiv \delta \int_V |\nabla\Phi|^2 dV = 0, \quad (6)$$

gives Laplace PDE (5) as the Euler-Lagrangian equation for a Φ that renders E an extremum but also an absolute minimum. That a minimum in $E \neq 0$ exists would thus prove that a solution to the Laplace boundary value problem exist. There was much excitement with Riemann’s proof but enthusiasm waned away when Weierstrass rejected the proof, realizing that an extremization must be defined over a specific function space and a distinction must be made between the existence of a minimum value of E and whether a function exists in that space that realizes the minimum value. In other words, can the extremization be carried out over a compact space in the sense of a Cauchy complete sequence? The latter is meant that a convergent sequence in a space has its limit also found in the space. The analogy is clear by the example of the open interval $0 < x < 1$, that a sequence of numbers in the interval converging to the end point 1 is not Cauchy complete. Weierstrass’ criticism of Riemann and a general understanding of Cauchy complete and incomplete function spaces had to wait for Hilbert’s work some 50 years after Riemann’s thesis; *What is mathematics?*, R. Courant & H. Robbins, 1969, 14th Printing, Oxford University Press.

Most spaces of continuous functions are not Cauchy complete, a fundamental nature of variational problems; Page 173, *Methods of Mathematical Physics*, Vol. 1, R. Courant & D. Hilbert, 1966, Interscience). The 1972 ApJ paper of Parker on Topological Dissipation is a physically motivated proof that for most field topology T , there is no (continuous) equilibrium field solutions. Physically we know there has to be an equilibrium solution for every field in V because the hydromagnetic system in domain V has a finite free energy bounded below by the energy E_{pot} of the potential field \mathbf{B}_{pot} . The field constructed to evolve along a path of monotonically-decreasing magnetic energy in phase space $\mathcal{S}(T)$ must physically reach equilibrium as free energy runs out at a lower bound $E_{\text{min}} \geq E_{\text{pot}}$. The minimum energy E_{min} is not in question. If that minimum energy state cannot be realized by a continuous solution to the variational problem (3), then the evolution must end up with a discontinuous equilibrium state. Not any discontinuity is admissible, but those TDs whose CSs together with the continuous part of the force-free current density by Ampere’s law give a field of topology T .

One may say intuitively that the contraction of function space from \mathcal{S}_{all} to its subspace $\mathcal{S}(T)$ can lose the property of Cauchy completeness in the variational problem. Courant & Robbins have an excellent analogy. Consider two points P and Q on the x axis. Seeking a curve on the Cartesian plane connecting P and Q of the shortest curve length yields the extremum realized by the straight line from P to Q . If we restrict the curves connecting P to Q to the subspace of curves that leave from P and arrive at Q both at right angle to the x axis, the extremization for the minimum curve length yields in the limit the same straight line from P to Q , but this limit is never realized in the reduced space of curves leaving P and arriving Q at right angle to the x axis.

Rewrite the Euler-Lagrangian equations (4) for a continuous force-free field in the equivalent form

$$\nabla \times \mathbf{B}_{ff} = \alpha \mathbf{B}_{ff}; \quad \mathbf{B}_{ff} \cdot \nabla \alpha = 0, \quad (7)$$

introducing the proportionality function α for the force-free field-aligned current density. The force-free equations for an extremum equilibrium field embedded with TDs is described by a generalized equilibrium equation

$$\nabla \times \mathbf{B}_{ff} = \alpha \mathbf{B}_{ff} + \sum_{\sigma} \mathbf{I}(\sigma) \delta(\sigma); \quad \mathbf{B}_{ff} \cdot \nabla \alpha = 0, \quad (8)$$

where the Dirac delta-function $\delta(\sigma)$ describes discrete currents of intensities $\mathbf{I}(\sigma)$ flow in TD surfaces σ whose shapes are determined as free boundary problem, coupled to the global field with continuity of the magnetic pressure B^2 across these TD surfaces. These are formidable, highly nonlinear and nonlocal boundary value problems. The 2D Cartesian case is an exact example of the complex 3D case; first constructed in the Appendix of Low (1989).

3. A 2D Cartesian Case. Consider the potential field \mathbf{B}_{pot} in the $y - z$ plane in Figure 1 with 3 field reversal lines on the base $z = 0$, symmetric about the vertical z axis where an X -type magnetic neutral point is located associated with field lines closing over the origin $y = z = 0$. The perfectly conducting fluid domain V is identified with the unbounded atmosphere $z > 0$ and boundary ∂V with the atmospheric base $z = 0$ where the normal flux distribution $B_z = B_n(\partial V)$ is fixed. The space \mathcal{S}_{all} of all the continuous fields in $z > 0$ share the same $B_z = B_n(\partial V)$ at $z = 0$ but differ in field topology T depending the specific connectivity of the field lines of the multipolar field in $z > 0$. The potential field \mathbf{B}_{pot} in Figure 1 has a specific topology T_{pot} . Figure 2 displays a field with a field line connectivity distinct from the potential field, the former, in particular, having no X -type neutral point along the z -axis and no field lines closing over the origin $y = z = 0$, i.e., the lines emerging from $z = 0$ on the two immediate sides of the origin connect away from the origin in their respective sides. This field denoted by \mathbf{B}_{neq} is not in equilibrium, with a topology T_{neq} distinct from T_{pot} . There is no way to deform \mathbf{B}_{neq} continuously into \mathbf{B}_{pot} . However, a

continuum of fields all continuously deformed from \mathbf{B}_{neq} generates the subspace $\mathcal{S}(T_{\text{neq}})$. Thus, by calculating the extremum in magnetic energy over the space $\mathcal{S}(T_{\text{neq}})$ we may seek the equilibrium of the field \mathbf{B}_{neq} with topology T_{neq} .

Now, the solenoidal 2D Cartesian field may be expressed in terms of a magnetic stream function A :

$$\mathbf{B} = \left[0, \frac{\partial A}{\partial z}, -\frac{\partial A}{\partial y} \right], \quad (9)$$

$$\nabla \times \mathbf{B} = \left[-\nabla^2 A, 0, 0 \right], \quad (10)$$

showing that the current density is always perpendicular to the field in the $y - z$ plane. It follows that, if the field is force-free, its current density must vanish and it must thus be the unique potential field \mathbf{B}_{pot} fixed by the boundary flux distribution on $z = 0$, in Figure 1. Thus, seeking the extremum:

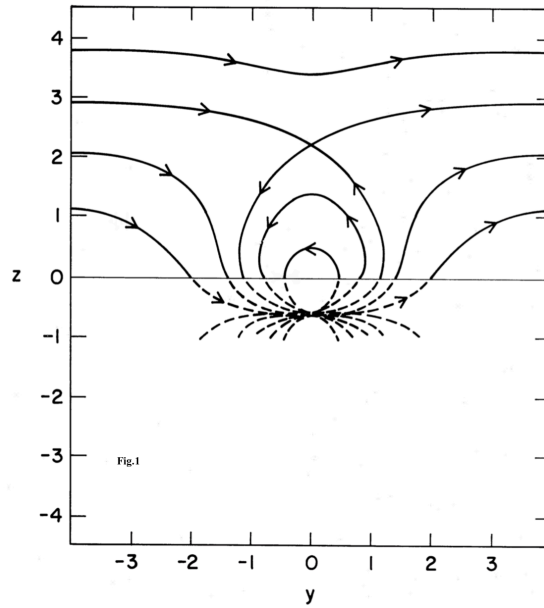
$$\delta E \equiv \delta \int_{z>0} B^2 dydz \equiv \delta \int_{z>0} \left[\left(\frac{\partial A}{\partial y} \right)^2 + \left(\frac{\partial A}{\partial z} \right)^2 \right] dydz = 0, \quad (11)$$

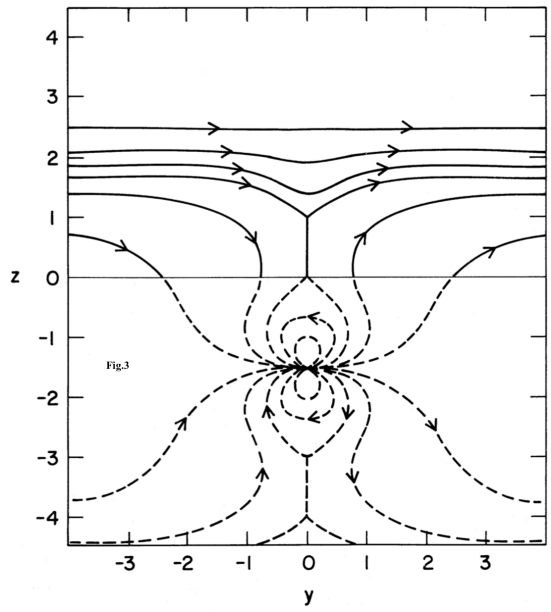
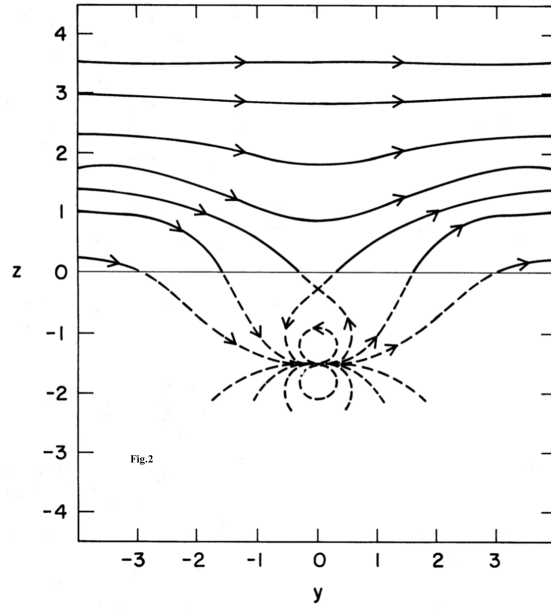
over any subspace $\mathcal{S}(T)$ and the fixed boundary flux distribution on $z = 0$ will yield no continuous force-free field except for the one case of $T = T_{\text{pot}}$. For *all* $T \neq T_{\text{pot}}$, the extremum in E is found in a field which is everywhere potential except for the presence of a TD where the tangential field reverses discontinuously across so that the magnetic pressure B^2 is the same on the two sides of the TD for force-balance.

The fixed boundary flux distribution on $z = 0$ being symmetrical in y in the case of $T = T_{\text{neq}}$, the field \mathbf{B}_{neq} can assume the equilibrium state shown in Figure 3, where the TD is of a finite length extending from the origin vertically and symmetrically placed along the z axis. Under the "frozen in" condition of perfect conductivity, the non-equilibrium field can decrease its free energy monotonically by having the field on the two sides of the z axis just above the origin push inward to eject upward the field crossing the z axis. The two side fields can then come into direct contact to form the TD as shown.

What seems to be a general feature to be found under general circumstances is the expulsion of a third flux subsystem from out between two converging flux subsystems, initially spatially apart, to come into contact and form the TD. The converging motion of the two subsystems is driven by the Lorentz forces in the non-equilibrium field \mathbf{B}_{neq} during an energy-decreasing evolution. Of course, in fully dynamical evolution, the fluid if not inertia-less may have sufficient kinetic energy to bring about the converging pair of subsystems of flux, but that is quite irrelevant to the fact that in the field dominated evolution under perfect conduction, the converging drive of the Lorentz force is a monotonically energy-decreasing

action, and the TD thus form inevitably. Such interactions involving three or more flux systems are prevalent in 3D fields of complex topologies, as demonstrated by the wealth of illustrative theoretical examples in the above cited Book of Parker.





Conversations with Eugene N. Parker¹

III. Magnetic Field Topology

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The meaning of magnetic field topology described in Conversations I.3 and II.1 is illustrated with fields in an upright cylindrical domain with rigid boundary walls. The rigid-boundary conditions are only cited in the previous Conversations, but are now derived for a direct understanding of how field topology is invariantly trapped by the rigid walls. The meaning so illustrated is then related to the Parker two-plate topological boundary value problem.

1. Induction Equation and Boundary Conditions. Consider a magnetic field \mathbf{B} in the cylindrical domain $V : R < R_0, -L < z < L$, using cylindrical coordinates, containing a perfect fluid conductor, the field being tangential along $R = R_0$ with prescribed, positive definite flux-distributions $B_z = F_{\pm L}(R, \varphi)$ at $z = \pm L$. For simplicity we take \mathbf{B} to be uni-directional, entering at $z = -L$ and leaving at $z = L$, equal total fluxes at these two boundaries.

The field may be deformed by any continuous fluid velocity \mathbf{v} according to the induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}), \quad (1)$$

subject to 2 distinct sets of rigid-boundary conditions. The tangential component of the electric field $\mathbf{E} = \frac{1}{c} \mathbf{v} \times \mathbf{B}$ must be continuous across any surface and must, therefore, vanish at the perfectly conducting wall;

$$\hat{\mathbf{n}} \times (\mathbf{v} \times \mathbf{B}) = (\hat{\mathbf{n}} \cdot \mathbf{B}) \mathbf{v} - (\hat{\mathbf{n}} \cdot \mathbf{v}) \mathbf{B} = 0, \quad (2)$$

$\hat{\mathbf{n}}$ being the boundary unit normal. Applying the rigid-boundary condition, $v_n = \hat{\mathbf{n}} \cdot \mathbf{v} = 0$, we then have

$$B_n \mathbf{v} = 0. \quad (3)$$

At $z = \pm L$ where $B_n = B_z = F_{\pm L}(R, \varphi) > 0$ we have the vector boundary conditions

$$z = \pm L, \quad \mathbf{v} = 0, \quad (4)$$

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whereas $B_n = 0$ at the side boundary $R = R_0$, and condition (3) is always satisfied where we have a less restrictive condition on the velocity

$$R = R_0, \quad v_n = 0, \quad (5)$$

i.e., the tangential component of \mathbf{v} along the side boundary is determined by the interior flow. By boundary conditions (4), the flux distributions $F_{\pm L}(R, \varphi)$ are constant in time.

2. Case of Uniform Field. Consider $\mathbf{B} = B_0 \hat{\mathbf{z}}$ and $\mathbf{v} = 0$ with $F_{\pm L} = B_0$, a constant. Henceforth we take the fluid in V to be inertia-less and field equilibrium is defined by the vanishing of the Lorentz force

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = 0, \quad (6)$$

trivially satisfied by the uniform field. Subject to the boundary conditions (4) and (5), any velocity can be kinematically imposed on the uniform field, continuously deforming the field forward in time into a non-equilibrium states in V sharing the invariant boundary flux $B_z = F_{\pm L} = B_0$ at $z = \pm L$. Of course, reversing the field deformation will restore the non-equilibrium state back to its equilibrium uniform state.

In both forward and reversed deformations, two properties are invariant under boundary condition (4), namely, the boundary flux distribution $F_{\pm L} = B_0$ and the identity map \mathcal{M} of the foot-points of $z = -L$ onto the foot-points on $z = L$. Introduce the Cartesian coordinates, $\mathbf{X} = (x, y, z) = (R \cos \varphi, R \sin \varphi, z)$ and denote by $\mathbf{X}_1 = (\xi_1, \zeta_1, -L)$ and $\mathbf{X}_2 = (\xi_2, \zeta_2, L)$, the coordinates of the pair of foot-points of a field line on $z = \pm L$, respectively. The general foot-point map may be expressed as $\mathcal{M} : (\xi_1, \zeta_1) \rightarrow (\xi_2, \zeta_2)$. In the case of the uniform field, its foot-points on $z = \pm L$ are related by the identity foot-point map $\mathcal{I} : (\xi_1, \zeta_1) \equiv (\xi_2, \zeta_2)$.

The fundamental property of the induction equation (1) is the conservation of magnetic flux across every fluid surface in its Lagrangian motion in V . This frozen-in condition defines a specific relation between each parcel of fluid to the field in whatever state accessible by the induction equation. Flux conservation is not an Eulerian property, in the sense of expressing all physical variables as functions of spacetime with no need to know where each specific parcel of fluid happens to be located at a given time. Nevertheless, the Lagrangian knowledge of where the fluid parcels are at any given time carries information of the conserved fluxes across its fluid surfaces invariant for all time. That invariant information defined in the initial uniform field is retrievable by the history of the imposed velocity \mathbf{v} that has moved the fluid around and deformed the field from its initial uniform state. The invariance of $B_z = F_{\pm L} = B_0$ at $z = \pm L$ is just the consequence of the boundary fluid at $z = \pm L$ not moving, complying with Maxwell's equation on the electric field. The same boundary condition on the velocity and the frozen in condition throughout V demand that the connectivity of boundary foot-points by individual field lines cannot change.

Clearly, the preceding are only two of the continuum of the invariant properties of the field all expressible in terms of the flux surfaces and the field lines as intersections of flux surface. The invariant map \mathcal{M} involves individual field lines. Other invariants involve 2 or more field lines. For example, a pair of anchored field lines subtend a fluid surface that has an invariant total flux across it. To see the additional invariants needed to characterize the non-equilibrium field deformed from the uniform field, note that the identity map $\mathcal{I} : (\xi_1, \zeta_1) \equiv (\xi_2, \zeta_2)$ of the uniform field is consistent with an infinity of states having intertwining field lines in V .

The preceding analysis has thus revealed a fundamental point of the Parker Magneto-static Theorem. Unlike the classical PDE boundary value problem (BVP), we are dealing with a system of *nonlocal* PDE-integral equations comprising the force free equation (6) coupled to integrals expressing the frozen-in invariants. From this perspective, we expect over-determination as a rule and this expectation takes away the surprise that the continuous force-free fields as a solution to the PDE-integral BVP are, in general, non-existent.

The invariants all have their origin in flux conservation on each and any fluid surface. In other words, the conservation law is expressed as properties of magnetic surfaces and volumes like the flux tubes. The textbook definition is that topology comprises all those properties of a given set of geometric objects that are invariant under all continuous deformations of the objects. Thus, the said conservation-law invariance are the topological invariants governing the deformation of flux surfaces as the geometric objects, under the frozen-in condition.

3. Frictional Relaxation. Given any non-equilibrium field $\mathbf{B}_{\text{initial}}$ in V , a simple direct way of determining an equilibrium state available to it is to kinematically construct an explicit velocity governed by equations (1), (4) and (5), that takes the initial field $\mathbf{B}_{\text{initial}}$ to an extremum in the field’s total energy. This works in a conceptually simplest way if the equilibrium state sought after is a local minimum in magnetic energy. We then find this energy minimum by constructing a velocity that is everywhere in the same direction of the Lorentz force at all time. The field will be doing work to drain away its total energy persistently, and there is only a finite free energy to be removed.

As an aside, a popular model of the frictional relaxation is based on setting

$$\mathbf{v} = \rho_0 (\nabla \times \mathbf{B}) \times \mathbf{B}, \tag{7}$$

where ρ_0 is a free constant parameter. It was not appreciated that during the relaxation before the field becomes force-free everywhere, the Lorentz force of the evolving field is not zero all the way to the boundary. That is, the velocity given by equation (7) violates the boundary conditions (4) and (5). Many numerical studies claiming against the Parker theorem are based on this fatally flawed relaxation model (Low 2014). An algebraically defined

velocity has no boundary conditions to satisfy, only PDEs go with boundary conditions. An example of the latter, not explored in the literature, is based on balancing the Lorentz force with the viscous force,

$$\nabla^2 \mathbf{v} = \rho_0 (\nabla \times \mathbf{B}) \times \mathbf{B}, \quad (8)$$

with a positive constant ρ_0 . In this case we have a physically interpretable model. Note that the total magnetic energy will be monotonically decreasing in time because viscosity always drains energy under boundary conditions (4) and (5).

From the mathematical point of view, frictional relaxation is just one of many mathematical methods of searching among all the continuous fields admissible in V a particular field that is an extremum in magnetic energy. The frictional relaxation model is physically interpretable, but any mathematical optimization algorithm would do the same job. From here we arrive at recognizing that the force-free field is the variational extremum

$$\delta \int_V B^2 dV = 0, \quad (9)$$

in the vector function space $\mathcal{S}(F_{\pm L}, T)$ of continuous solenoidal fields admissible in V sharing the same $F_{\pm L}$ and all the topological invariants denoted by T .

4. The 1972 Parker Theorem in context of the above cylindrical domain V asks for the equilibrium state for a general non-equilibrium field created from the uniform field in V by the following construction. Starting with the uniform field, relax the rigidity of the lower boundary $z = -L$ where the foot-points are tangentially and continuously displaced. The displacement is rotational but incompressible, the latter to preserve the boundary flux distributions $B_z = F_{\pm L} = B_0$ at $z = \pm L$. This boundary displacement is continue into the interior of V , deforming the interior field, initially uniform, into a non-uniform non-equilibrium $\mathbf{B}_{\text{initial}}$. Even with rotational foot-point displacements, it is possible for an intended tailoring of the displacement to end up with a $\mathbf{B}_{\text{initial}}$ possessing the same topological invariants as the initial uniform field, despite the tangled state of $\mathbf{B}_{\text{initial}}$, but, in general, the state $\mathbf{B}_{\text{initial}}$ created has a different set of topological invariants, the case we henceforth consider here. The boundary distribution $B_z = F_{\pm L} = B_0$ at $z = \pm L$ is preserved, but the foot-point mapping $\mathcal{M} : (\xi_1, \zeta_1) \rightarrow (\xi_2, \zeta_2)$ is no longer the identity map of the initial uniform field.

Now restore the rigidity of boundary $z = -L$ and boundary conditions (4) and (5) apply. By its construction, no continuous deformation in V can deform $\mathbf{B}_{\text{initial}}$ into the uniform field. The vector function space $\mathcal{S}(F_{\pm L}, T)$ can be generated by a direct construction that deforms non-equilibrium $\mathcal{S}(F_{\pm L}, T)$ to all the fields in that space by all continuous velocity subject to equations (1), (4) and (5). The uniform field is not to be found in $\mathcal{S}(F_{\pm L}, T)$ and the question

is whether a continuous force-free field does exist in $\mathcal{S}(F_{\pm L}, T)$. The Parker Theorem asserts that, in general, an arbitrarily created $\mathcal{S}(F_{\pm L}, T)$ may not have a continuous equilibrium state to be found in $\mathcal{S}(F_{\pm L}, T)$.

The boundary conditions (4) and (5) are the at the crux of the issues. The foot-point displacements creating $\mathbf{B}_{\text{initial}}$ can, of course, undo the creation by simply imposing the reversed foot-point displacements under a continued relaxation of the the boundary $z = -L$ from rigidity. The essential point of the Parker problem is that after the field $\mathbf{B}_{\text{initial}}$ has been created topologically not the same as the uniform field, it is now locked in its acquired topology by the restored rigidity of $z = -L$. There is then no way for any interior velocity to return this created field to the uniform field.

5. Axisymmetric PDE-Integral Equation Consider the axisymmetric ($\frac{\partial}{\partial \varphi} \equiv 0$), field

$$\mathbf{B} = \frac{1}{R} \left[-\frac{\partial H}{\partial z} \hat{\mathbf{R}} + K \hat{\boldsymbol{\varphi}} + \frac{\partial H}{\partial R} \hat{\mathbf{z}} \right], \quad (10)$$

lying on axisymmetric flux surfaces of constant $H(R, z)$ with an azimuthal field component K/R . Setting $H = \frac{1}{2} B_0 R^2$, $K = 0$ gives the uniform field $\mathbf{B} = B_0 \hat{\mathbf{z}}$ with the identity foot-point mapping between $z = \pm L$.

Let us relax the rigidity of $z = -L$ where the foot-points of the uniform field is then tangentially displaced strictly in the azimuthal direction, the boundary displacement suitably continued into the fluid interior. For simplicity, the differentially rotating displacement is incompressible so that the symmetry about the axis $R = 0$ is preserved and the boundary flux distributions $B_z = B_0$ on $z = \pm L$ are unchanged, but the foot-point map \mathcal{M} is no longer the identity map of the uniform field. We now restore this $z = -L$ boundary to rigidity and each field line then connects a foot-point at (R, φ, L) on $z = L$ to its other foot-point $(R, \varphi + \Delta\varphi(R), -L)$ on $z = -L$ where $\Delta\varphi(R)$ is the net angular displacement imposed on the foot-points of the boundary $z = -L$. In other words, now with foot points rigidly anchored under boundary conditions (4) and (5), the field (H, K) evolve in V preserving the foot-point map \mathcal{M} connecting the foot-points on each circle of radius R on $z = L$ to a corresponding circle of foot-points on $z = -L$ of the same radius but rotated by $\Delta\varphi(R)$. We require $\Delta\varphi = 0$ at $R = 0$ for geometric reason but we also demand that it vanishes at $R = R_0$, a feature useful for a purpose later.

To relate (H, K) in any of the deformed states in V to the invariant map \mathcal{M} , we integrate

$$\frac{Rd\varphi}{B_\varphi} = \frac{ds}{[B_R^2 + B_z^2]^{1/2}}, \quad (11)$$

for the net angular displacement $\Delta\varphi$, a path integral with path length s along the full length of the projection of each field line of constant $H(R, z)$, from a foot-point on $z = -L$ to the

other foot-point on $z = L$. We thus obtain

$$\Delta\varphi(R) = \int_H \frac{K(R, Z)}{R} \left[\left(\frac{\partial H}{\partial R} \right)^2 + \left(\frac{\partial H}{\partial z} \right)^2 \right]^{-1/2} ds, \quad (12)$$

attaching the subscript H to the integral to indicate an integration along a curve of constant H on the $R - z$ plane that intersects $z = \pm L$ at radius R appearing as the argument of the prescribed function $\Delta\varphi$. We now have the Parker topological problem of seeking a force-free field subject to boundary conditions $B_z = B_0$ at $z = \pm L$, $B_R = 0$ at $R = R_0$ and topologically constrained to satisfy integral equation (12) for an arbitrarily prescribed $\Delta\varphi$.

The field (10) has the current density

$$\nabla \times \mathbf{B} = \frac{1}{R} \left[-\frac{\partial K}{\partial z} \hat{\mathbf{R}} - \mathcal{L}H \hat{\boldsymbol{\phi}} + \frac{\partial K}{\partial R} \hat{\mathbf{z}} \right], \quad (13)$$

$$\mathcal{L} = R \frac{\partial}{\partial R} \left[\frac{1}{R} \frac{\partial}{\partial R} \right] + \frac{\partial^2}{\partial z^2}. \quad (14)$$

Substitution into the force-free equation (6), we obtain

$$K(R, z) = K[H(R, z)], \quad (15)$$

$$R \frac{\partial}{\partial R} \left[\frac{1}{R} \frac{\partial H}{\partial R} \right] + \frac{\partial^2 H}{\partial z^2} + \frac{1}{2} \frac{d}{dH} (K^2) = 0, \quad (16)$$

stipulating that the $RB_\varphi = K(R, z)$ must be constant along a field line of constant H , the profile $K(H)$ then defines the PDE (16) to be solved as a boundary value problem for H in terms of which K is then defined. For the above uniform field and its foot-point displaced deformed field, the boundary conditions are, $H = \frac{1}{2}B_0R^2$ at $z = \pm L$ and $H = \frac{1}{2}B_0R_0^2$ at $R = R_0$. In the Parker problem, it is the field topology that is prescribed, by explicitly specifying the function $\Delta\varphi$, from which $K(H)$ must be determined by integral equation (12) coupled to PDE (16).

We may invert the relationship $H(R, z) = H_0$, a constant, to express the integrand in integral equation (12) as a strict function of z and H along the path of integration. Therefore, taking the integral with z as the integration variable instead of path length s , the corresponding integration limits are $z = \pm L$, so that the definite integral is a strict function of H . At the integration limits $z = \pm L$, the boundary values of $H = \frac{1}{2}B_0R^2$ allows the profile function $\Delta\varphi(R)$ to be formally translated into a function of H , i.e., $\Delta\varphi(R) = \Delta\varphi\left(\sqrt{\frac{2H}{B_0}}\right)$. Thus, integral equation (12) defines the functional form of $K(H)$:

$$K(H) = \Delta\varphi\left(\sqrt{\frac{2H}{B_0}}\right) \left(\int_H \left[\left(\frac{\partial H}{\partial R} \right)^2 + \left(\frac{\partial H}{\partial z} \right)^2 \right]^{-1/2} \frac{ds}{R} \right)^{-1}, \quad (17)$$

where we have assured ourselves that the definite integral is a strict function of H . The governing equation for H is thus a PDE-integral equation,

$$R \frac{\partial}{\partial R} \left[\frac{1}{R} \frac{\partial H}{\partial R} \right] + \frac{\partial^2 H}{\partial z^2} + \frac{1}{2} \frac{d}{dH} [\Delta\varphi^2 \left(\sqrt{\frac{2H}{B_0}} \right) \left(\int_H \left[\left(\frac{\partial H}{\partial R} \right)^2 + \left(\frac{\partial H}{\partial z} \right)^2 \right]^{-1/2} \frac{ds}{R} \right)^{-2}] = 0, \quad (18)$$

subject to the boundary conditions on H , a formidable boundary value problem with little known of its mathematical properties. Clearly, there are continuous solutions. Instead of the Parker problem, we may prescribe $K(H)$ explicitly and solve PDE (16) subject to the boundary conditions on H . A solution so constructed defines a profile $\Delta\varphi(H)$ to identify *a posteriori* the foot-point displacement consistent with the solution obtained. The Parker theorem deals with the problem starting with an arbitrarily chosen $\Delta\varphi(H)$ with the claim that, in general, no continuous solution exists except for a restricted subspace of the prescribed $\Delta\Phi(H)$. In other words, all the continuous force-free fields constructed by prescribing $K(H)$ have $\Delta\Phi(H)$ which constitute only a subset of all the $\Delta\Phi(H)$ one can reasonably physically prescribe. Is the preceding statement correct?

The above topological problem posed by an arbitrarily prescribed $\Delta\Phi(H)$ is generally intractable by its nonlinear and nonlocal nature, although the formal derivation of the governing PDE-integral equation is straightforward for the axisymmetric case. An ignorable coordinate severely suppresses a degree of freedom and its whole continuum of topological invariants that takes the Parker problem to a different realm of intractability. Numerical relaxation of a field to a topologically defined initial field in V , in the sense of the Parker theorem, has much to offer not just for understanding the theorem but also for the intrinsic challenge of 3D numerical simulation. Numerical studies of this kind in recent decades had suffered from both physicists and numerical mathematicians who generally are plainly no good in their work, it seems.

6. A Non-Asymmetric Application. We end with the Parker two-plate problem asking for the force-free state of a non-equilibrium field anchored rigidly at the two plates $z = \pm L$ where the normal field is uniform. Take the two-plate initial uniform field anchored to the two plates. Then, with the foot-point displacement $\Delta\varphi(R)$ satisfying $\Delta\varphi(R_0) = 0$, the foot-point displaced cylindrical field may be inserted between the plates to replace the initial uniform field equal in volume to the cylindrical domain. Indeed, this can be done with many cylindrical replacements, each with a different $\Delta\varphi(R)$ and cylindrical radius R_0 . In each case, the field continues into the original uniform field by virtue of $\Delta\varphi(R_0) = 0$.

The interesting point is that two such cylindrical fields may be placed in contact along a vertical line. These are non-equilibrium fields. In each cylinder, a net current flows in one

direction around the axis of symmetry and a net opposite current flows in a layer bordering the cylindrical boundary. Opposite currents repel and the two cylindrical fields in a line contact will expand outward, pushing the two wedges of external uniform fields away and pushing into each other to squeeze the line contact into a surface contact, a TD or a complex of TDs seemingly inevitable; most interesting is the case of the two cylindrical fields having greatly different $\Delta\varphi(R)$ and R_0 , Here we may see the dual effects at work, that of the optical analogy and the packing of flux tubes with Y-shaped separatrices. Numerical work is best done with forethought, and here is a promising 3D simulation worthy of consideration.

Conversations with Eugene N. Parker¹

IV. Boundary Singularity

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In the Parker Problem for a field anchored to the rigid boundary, any spontaneous TD (or CS) forming in the domain will extend along a flux surface to the boundary where the normal flux distribution, given to be continuous, remains invariant. Here we examine the equilibrium intersection of the TD with the boundary to derive the inevitability of a strong current singularity first encountered by Low & Wolfson (see page 578 in 1988, ApJ 324, 574-581). Such a singularity, admissible for a perfect conductor, must dissipate for a near-perfect conductor, thus forcing a resistive change in the boundary normal flux distribution.

1. Formulation of Problem. Consider a 2D Cartesian field

$$\mathbf{B} = \left(\frac{\partial A}{\partial z}, -\frac{\partial A}{\partial y} \right), \quad (1)$$

anchored rigidly at horizontal lower boundary $z = 0$. Setting $A = -B_0 y$ gives $\mathbf{B} = B_0 \hat{z}$, a uniform field perpendicular to this boundary. For specificity, we may take this field to be anchored at some upper boundary $z = L$, but this detail is not essential for our consideration.

Take the compressible fluid in $0 < z < L$ to be a highly anisotropic thermal conductor, perfect conduction along the field, zero conduction across the field. Initially the fluid has a uniform temperature T_0 maintained by the rigid boundaries serving as a thermal reservoir. Thus the pressure p_0 is uniform and exerts no force on the uniform field. Now let the temperature of the boundary reservoirs $y > 0$ go to zero whereas T_0 is maintained in the boundary reservoirs $y < 0$. The non-equilibrium field thus created must relax to an equilibrium state in which the field is everywhere potential matching the boundary condition

$$z = 0, L, \quad B_z = B_0, \quad (2)$$

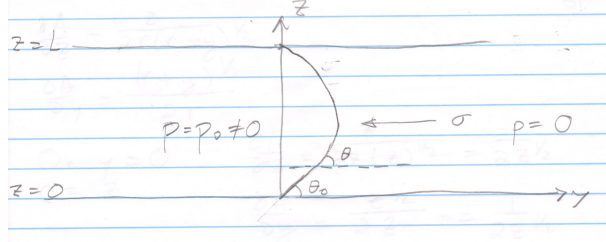
but inevitably contains a CS located along a curved field line σ connecting the foot-points $(0, 0)$ and $(0, L)$, σ having been deformed, under the frozen-in condition, from the straight field line that connects these two foot-points in the uniform-field initial state; see Fig. 1. With the original uniform pressure removed from the right half space $y > 0$, σ bulges to the

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right. Let \mathbf{B}_1 and \mathbf{B}_2 respectively denote the potential fields to the left and right of σ whose shape is determined by the equilibrium condition

$$\text{across } \sigma, \quad B_1^2 + 8\pi p_0 = B_2^2. \quad (3)$$

That σ must be a TD is straightforward whereas the equilibrium nature of the intersection of this CS with boundaries $z = 0, L$ has a subtlety. We concentrate on $z = 0$.



2. Apparent Paradox & Resolution. Denote by θ the horizontal inclination of \mathbf{B}_1 and \mathbf{B}_2 along σ , sketched in Fig. 1, with $\theta = \theta_0$ being the inclination at the $z = 0$ foot-point. On the two sides along σ , we have the two fields

$$\mathbf{B}_1 = (B_1 \cos \theta, B_1 \sin \theta), \quad (4)$$

$$\mathbf{B}_2 = (B_2 \cos \theta, B_2 \sin \theta). \quad (5)$$

Taking these two fields along σ to the foot-point at $z = 0$, we get $B_z = B_1 \sin \theta_0$ and $B_z = B_2 \sin \theta_0$ on the two sides of σ . By equation (3) $B_1 \neq B_2$ and we discover that B_z on $z = 0$ is discontinuous across the σ foot-point, in violation of boundary condition (2) if $\theta_0 \neq 0$. However, if we set $\theta_0 = 0$ implying that field-line σ threads $z = 0$ horizontally, the apparent paradox deepens with $B_z = 0$ at the σ foot-point, still in violation of boundary condition (2). Moreover, with $\theta_0 = 0$, we have $\mathbf{B}_1 = B_1 \hat{y}$ and $\mathbf{B}_2 = B_2 \hat{y}$, resulting in two unequal horizontal fields on the two sides of the σ foot-point, which makes no geometric sense.

These contradictions are resolved by realizing that the rigidity of $z = 0$ forbids any change in its normal flux distribution and a simple CS cannot exist to balance the jump in fluid pressure. The nonzero pressure p_0 simply pushes σ outwards to effect a tangential contact with $z = 0$ in order to have the reaction force of the rigid boundary balance the pressure jump. This forcing results in such an extreme stressing of the field that the σ foot-point is a point infinity in the field intensity. That is, along σ , both field intensities $B_1 \rightarrow \infty$ and $B_2 \rightarrow \infty$ with $\theta \rightarrow 0$ upon arriving at the foot-points. This limit is constrained to ensure that (i) $B_1 \sin \theta \rightarrow B_z$ and $B_2 \sin \theta \rightarrow B_z$ under boundary condition (2) and (ii) $B_1^2 \rightarrow \infty$ and $B_2^2 \rightarrow \infty$ with a fixed off-set $8\pi p_0$ between them so that the TD equilibrium

condition (3) is satisfied. Additionally, the point singularities must be integrable to give finite magnetic energies.

This point-singularity property was encountered first in a different context, the intersection of a TD separating force-free magnetic fields (Low and Wolfson 1988). It is actually a line-singularity in general 3D fields, the line being the intersection of a TD flux surface with the boundary.

3. Two implications. If the fluid is taken rigorously perfectly conducting, the boundary foot-point singularity would form and persist permanently. For a near-ideal fluid with a small resistivity, the singularity would be approached only to result in resistive dissipation when field gradients become sufficiently steep. This topological dissipation of TDs as a source of heating in the corona may be expected to be most intense at the foot-points of the TDs.

Interesting to consider the TD boundary singularity in the Parker two-plate problem for a rigorously perfect conductor. Let us accept the van Ballegooijen nonlinear extrapolating to construct a force-free field from $z = -L$ to $z = L$ as an analog of the 2D initial value problem in pure vortex dynamics. In this construction, the magnetic state at $z = -L$ defines an initial 2D state, with the Cartesian coordinate z treated as time that takes the initial state to the final state at $z = L$. If the force-free field to be constructed has inevitable TDs, these TDs must extend to both boundaries. Therefore, the van B extrapolation must treat initial states at $z = -L$ with TD singularities, if force-free fields with inevitable TDs are to be constructed. van B was only interested in those constructions with continuous initial states. We may conclude that he would be constructing the subspace of everywhere continuous force-free fields, a subspace, of measure zero, of all the force-free fields with and without TDs.

In more general terms, the van B construction is an example of exhaustively prescribing α in the equivalent description of the force-free equation:

$$\nabla \times \mathbf{B} = \alpha \mathbf{B}, \tag{6}$$

$$\mathbf{B} \cdot \nabla \alpha = 0, \tag{7}$$

and generate all the continuous force-free fields in a domain V matching a given (fixed) distribution of the normal field component of \mathbf{B} at the boundary ∂V . Such a boundary value problem may generate some force-free field solutions for study, but prescribing α without a physical meaning for its form is hazardous, because not all α satisfying equation (7) are assured to have solutions to the boundary value problem.

Conversations with Eugene N. Parker¹

V. Special Classes of 2D & 3D Constant- α Force-Free Fields

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Conversation V here and Conversation VII to follow treat 2D and 3D constant- α force-free magnetic fields in a cylindrical domain, covering some properties not appreciated or found in the published literature. Outstanding problems still exist for this class of spatially continuous force-free fields, interesting with their own issues of existence and uniqueness of solutions to boundary value problems (BVPs).

The constant- α fields are especially interesting when examined in relation to the Parker (1972, 1994) Theorem on the equilibrium of a magnetic field in a perfect fluid conductor. The Theorem is built on the two basic properties of perfect conductivity:

- (i) The field possesses an invariant topology under the ideal induction equation, the field having a physical identity defined by its invariant topology.
- (ii) In the complete absence of resistive dissipation, current sheets (CSs) may form and persist for any length of time as magnetic tangential discontinuities (TDs).

In a conceptual departure from the traditional magnetostatic BVPs, property (i) poses the problem to construct the static equilibrium for a field as a function of its invariant topology, leading to the Theorem that the equilibrium field of an arbitrarily prescribed topology must generally embed TDs. The journal literature often made too much of a surprise about encountering the occurrences of TDs, a surprise only if we forget that property (ii) is physically admissible in the perfect fluid conductor.

There is a continuum-infinity of field topologies each compatible with the field in a spatially continuous equilibrium state embedding no TDs. Once recognized that TDs are physically admissible, it is clear that these continuous equilibrium states are only a subset of the greater continuum of general equilibrium states that includes states embedding TDs. The Parker Magnetostatic Problem thus seeks to construct an equilibrium field comprising a continuous part satisfying the equilibrium PDEs, that, in general, embeds TDs as weak solutions to the PDEs, the global field to be constructed subject to a given invariant topology. In the analysis here of the constant- α fields as the simplest forms of continuous equilibrium

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states without TDs, we will have occasions to visit the implications of the Theorem, page references cited herein to the Book *Spontaneous Current Sheets in Magnetic Fields*, E. N. Parker, 1994, Oxford University Press.

Here we analyze limited classes of 2D and 3D constant- α fields useful for visualizing topological properties of equilibrium fields by explicit solutions. Conversation VI has various thoughts about the Parker Theorem extensively covered in the Book. Conversation VII takes up the formulation and solution of the problem for the general 3D constant- α field, pointing out a longstanding fundamental flaw in many publications that was discovered and resolved by Low & Fang (2014).

1. The axisymmetric constant- α fields. From Conversation III on Field Topology,

$$\mathbf{B} = \frac{1}{R} \left[-\frac{\partial H}{\partial z} \hat{\mathbf{R}} + K \hat{\boldsymbol{\varphi}} + \frac{\partial H}{\partial R} \hat{\mathbf{z}} \right], \quad (1)$$

in cylindrical coordinates describes a axisymmetric solenoidal field lying on flux surfaces of constant $H(R, z)$ with an azimuthal component $B_\varphi = K/R$. The force-free equations

$$\nabla \times \mathbf{B} = \alpha \mathbf{B}, \quad (2)$$

$$\mathbf{B} \cdot \nabla \alpha = 0, \quad (3)$$

require

$$K(R, z) = K[H(R, z)], \quad (4)$$

$$\alpha = \frac{d}{dH} K(H), \quad (5)$$

$$R \frac{\partial}{\partial R} \left[\frac{1}{R} \frac{\partial H}{\partial R} \right] + \frac{\partial^2 H}{\partial z^2} + \alpha(H) K = 0, \quad (6)$$

a generally nonlinear PDE.

We concentrate on the linear constant- α fields:

$$K = \alpha H, \quad (7)$$

$$R \frac{\partial}{\partial R} \left[\frac{1}{R} \frac{\partial H}{\partial R} \right] + \frac{\partial^2 H}{\partial z^2} + \alpha^2 H = 0, \quad (8)$$

with the separable solution for a given α^2 ,

$$H_{\alpha, \lambda} = R J_1(\lambda R) \exp(i\kappa z), \quad \kappa^2 = \alpha^2 - \lambda^2, \quad (9)$$

with real wave numbers λ and κ , $i = \sqrt{-1}$, for the case of $\alpha^2 > \lambda^2$.

We consider the upright cylindrical domain $V : R < R_0, -L < z < L$ containing a field satisfying

$$\text{at } R = R_0, \quad B_R = 0, \quad (10)$$

$$\text{at } z = \pm L, \quad B_z = F(R). \quad (11)$$

For a field symmetric about the plane $z = 0$, we set

$$H_{\alpha,\lambda} = RJ_1(\lambda R) \cos(\kappa z), \quad \kappa^2 = \alpha^2 - \lambda^2, \quad (12)$$

$$\mathbf{B}_{\alpha,\lambda} = \kappa J_1(\lambda R) \sin(\kappa z) \hat{\mathbf{R}} + \alpha J_1(\lambda R) \cos(\kappa z) \hat{\varphi} + \lambda J_0(\lambda R) \cos(\kappa z) \hat{\mathbf{z}}, \quad (13)$$

noting that $\frac{1}{R} \frac{d}{dR} [RJ_1(\lambda R)] = \lambda J_0(\lambda R)$.

The flux function H is defined up to a free constant, so we take $H = 0$ on the field line along the cylindrical axis. For a field anchored to $z = \pm L$ with $F \neq 0$, the boundary conditions on \mathbf{B} are expressed in terms of H by

$$H(R, \pm L) = H_L(R) = 2\pi \int_0^R F(R) R dR, \quad (14)$$

$$H(R_0, z) \equiv H_0 = H_L(R_0) = 2\pi \int_0^{R_0} F(R) R dR, \quad (15)$$

where H_0 is the net flux across both $z = \pm L$. Equation (15) expresses the boundary condition $B_R = 0$ at $R = R_0$ by identifying this boundary with the $H = H_0$ flux surface.

With the magnetic foot-points of the field lines rigidly anchored to $z = \pm L$, two fields in V are topologically different if one may not be deformed continuously into the other under the ideal induction equation. If $B_z(R, \pm L) = F(R)$ is positive definite, then all field lines leave $z = -L$ to connect with $z = L$. The same $F(R)$ admits fields of different topologies in V , seen in the distinct connectivities between $z = \pm L$ depending on $B_\varphi = K/R \neq 0$ that gives each field line, lying on a constant- H flux surface, an angular displacement along the line. If $B_z(R, \pm L) = F(R)$ has both algebraic signs, the admissible field topologies are even richer, with sub-sets of field lines originating from and returning to the same z -boundary. In this case, field topologies may also be different by the different amounts of flux that connect the same z -boundary versus the amounts of flux connecting the $z = \pm L$ boundaries.

2. The Woltjer Wholly-Contained Fields. Fields with $F \equiv 0$ with $H_0 = H_L \equiv 0$ are wholly contained within the boundary, comprising $R = R_0$ and $z = \pm L$, that is a single flux surface $H = 0$. Boundary conditions (14) and (15) take the homogeneous forms

$$H(R, \pm L) \equiv 0, \quad (16)$$

$$H(R_0, z) \equiv 0, \quad (17)$$

posing an eigenvalue problem that restricts the constant α^2 to a spectrum of denumerable eigenvalues and corresponding eigenfunctions:

$$\alpha^2 = \alpha_{m,n}^2 = \lambda_m^2 + \kappa_n^2 > 0, \quad (18)$$

$$H_{m,n}(R, z) = RJ_1(\lambda_m R) \cos(\kappa_n z), \quad (19)$$

where $m = 1, 2, 3, \dots$, $n = 1, 2, 3, \dots$ respectively index the zeros of the Bessel and trigonometric functions, that is, $\lambda_m = \xi_m/R_0$; $J_1(\xi_m) = 0$ and $\kappa_n = (2n - 1)\frac{\pi}{2L}$; $\cos(\kappa_n L) = 0$.

These are the Woltjer states of an extremum in magnetic energy \mathcal{E} subject to a fixed value \mathcal{H}_0 of magnetic helicity \mathcal{H} :

$$\delta\mathcal{E} \equiv \delta \int_V B^2 dV = 0; \quad \mathcal{H} \equiv \int_V \mathbf{A} \cdot \mathbf{B} dV = \mathcal{H}_0, \quad (20)$$

where $\mathbf{B} = \nabla \times \mathbf{A}$, the magnetic potential \mathbf{A} defined up to a free gauge. By virtue of V being simply-connected and the wholly-contained \mathbf{B} being tangential everywhere at the domain boundary, the magnetic helicity \mathcal{H} is physically meaningful and independent of the free gauge. The Taylor (1974) theory of relaxation is based on the hypothesis that \mathcal{H} is approximately conserved during turbulent dissipation of current sheets in a highly conducting fluid. By this drainage of free magnetic energy, the evolving field tends towards the minimum- \mathcal{E} Woltjer force-free state with $\alpha = \alpha_{1,1}$, the lowest $m = 1, n = 1$ eigenvalue of the spectrum defined by \mathcal{H}_0 . Whereas \mathcal{H}_0 is a continuous parameter to be prescribed, the parameter α in the linear force-free PDE (6) is to be picked out of a discrete spectrum of eigenvalues via the extraneous physics of the Taylor theory.

Non-existence of a field solution for most values of α is, of course, a familiar feature of eigenvalue problems. But, the absence of solution for α not an eigenvalue has more than an incidental mathematical interest, in the context of the Parker Magnetostatic Theorem. Once we recognize that TDs are natural structures in a force-free field under perfect conductivity; see property (ii) in the Introduction, the Woltjer continuous fields are seen as a subset of the general continuum of force-free states of an even greater range of field topologies, including fields of all constant values of α that embed equilibrium TDs. The discrete Woltjer eigenvalues identify those special states in this general continuum of states that are everywhere spatially continuous in the absence of TDs.

3. Anchored Fields. Rewrite boundary conditions (14) and (15) tidily as

$$H(R, \pm L) = H_L(R), \quad (21)$$

$$H(R_0, z) = H_0, \quad (22)$$

for an anchored field with $B_z(R, \pm L) = F(R) = \frac{1}{R} \frac{dH_R}{dR} \neq 0$, the net flux $H_L(R_0) = H_0 \neq 0$ in general. The solutions to the problem posed by these inhomogeneous boundary conditions on

H may be expressed as a superposition of a set of orthogonal base functions, exploiting the linearity of the force-free PDE (8). In the construction of the superposition below, we again encounter the property that not all values of α yield a proper solution for a given boundary flux function $H_L(R)$. Moreover, if a particular value of α permits a solution to exist, the solution is not unique. The non-existence and non-uniqueness of solutions to boundary value problems, in the complementary cases of whether or not the governing boundary conditions are homogeneous, are general properties in the Sturm-Liouville theory of linear PDEs. It is instructive for our purpose here to construct the anchored-field solutions from first principles.

We first treat the case $H_L(R_0) = H_0 = 0$ and later account for the case $H_0 \neq 0$ by a simple extension. Using the orthogonality property

$$\int_0^{R_0} J_1(\lambda_m R) J_1(\lambda_{m'} R) R dR = \delta_{m,m'} \int_0^{R_0} [J_1(\lambda_m R)]^2 R dR, \quad (23)$$

$H_L(R)$ satisfying $H_L(0) = H_L(R_0) = 0$ may be decomposed into the sum

$$H_L(R) = \sum_{m=1}^{\infty} h_m R J_1(\lambda_m R), \quad (24)$$

$$h_m = \frac{\int_0^{R_0} H_L(R) J_1(\lambda_m R) dR}{\int_0^{R_0} [J_1(\lambda_m R)]^2 R dR}, \quad (25)$$

where integers m (and m') index the Bessel zeros $\xi_m = \lambda_m R_0$, i.e., $J_1(\xi_m) = 0$, and $\delta_{m,m'}$ is the Kronecker symbol. In principle, the boundary flux function $H_L(R)$ is a function to be prescribed arbitrarily subject to $H_L(0) = H_L(R_0)$, that is, the coefficients h_m may be arbitrarily prescribed to define $H_L(R)$. Whether an anchored-field solution exists that matches $H_L(R)$ at $z = \pm L$ depends on whether a corresponding linear superposition of base functions may be constructed for the solution.

For a fixed value of α^2 , consider the base functions $H_{\alpha,\lambda_m}(R, z)$, $m = 1, 2, 3, \dots$ respectively satisfying the inhomogeneous boundary conditions

$$H_{\alpha,\lambda_m}(R, \pm L) = R J_1(\lambda_m R). \quad (26)$$

Provided the given α^2 is not an eigenvalue of the wholly-contained field given by equation (18), which means that $\cos(\kappa L) \neq 0$, the base function takes one of two forms:

$$H_{\alpha,\lambda_m}(R, z) = R J_1(\lambda_m R) \frac{\cos(\kappa z)}{\cos(\kappa L)}, \quad \text{if } \kappa^2 = \alpha^2 - \lambda_m^2 \geq 0, \quad (27)$$

$$= R J_1(\lambda_m R) \frac{\cosh(kz)}{\cosh(kL)}, \quad \text{if } k^2 = \lambda_m^2 - \alpha^2 \geq 0, \quad (28)$$

in terms of which we have the linear superposition

$$H(R, z) = \sum_{m=1}^{\infty} h_m H_{\alpha, \lambda_m}(R, z) \quad (29)$$

as the anchored-field solution we set out to construct. Included in this class of solutions are the z -independent solutions for $\alpha^2 = \lambda_m^2$, for which $k = 0$ and $\kappa = 0$ in equations (27) and (28).

If α^2 coincides with an eigenvalue of the wholly-contained field, that is, $\kappa = \kappa_{n'} = (2n' - 1)\frac{\pi}{2L}$ and $\alpha^2 = \alpha_{m', n'}^2 = \lambda_{m'}^2 + \kappa_{n'}^2$ for a (m', n') integer-pair; see equation (18), then $\cos(\kappa L) = 0$ and there is no solution of the force-free PDE (8) matching inhomogeneous boundary condition (26). Here we encounter a denumerable infinity of values of $\alpha^2 = \alpha_{m', n'}^2$ for which the anchored-field solution does not exist, unless, in each instance of m' , $H_L(R)$ is not arbitrarily prescribed but is specially restricted to have $h_{m'} = 0$. For those boundary values $H_L(R)$ with $h_{m'} = 0$, The anchored-field solution $H(R, z)$ given by equation (29) with $\alpha^2 = \alpha_{m', n'}^2$ does exist, matching a continuum of boundary values $H_L(R)$ with $h_{m'} = 0$, but is not unique because if $H(R, z)$ is a solution, then so is $H(R, z) + c_0 H_{m', n'}(R, z)$, where c_0 is an arbitrary constant.

The anchored-field (29) constructed under the conditions $H_L(0) = H_L(R_0) = 0$ carries no net flux along the length of the cylindrical domain. If $\alpha^2 \neq \lambda_m^2$, $m = 1, 2, 3, \dots$, we may generalize solution (29) to include a linearly superposed term, that is independent of z , to account for a net flux $H_0 \neq 0$ connecting $z = \pm L$,

$$H(R, z) = H_0 \frac{RJ_1(\alpha R)}{RJ_1(\alpha R_0)} + \sum_{m=1}^{\infty} h_m H_{\alpha, \lambda_m}(R, z), \quad (30)$$

to complete the construction of the anchored fields .

4. The 1972 Parker Magnetostatic Problem treats thin tubes of internally twisted flux of a width d anchored at two infinite plates $z = \pm L$, the field described by a power series in $\epsilon = d/L \ll 1$. It is shown that force-equilibrium (the case of force-free state in the analysis here) requires the field in the main length of the tube, away from boundary-layer structures abutting $z = \pm L$, must be invariant in z to all orders of the series in the limit of $\epsilon \rightarrow 0$.

Let us recover this asymptotic z -invariance in the anchored field (30). Suppose the net flux H_0 is the dominant contribution in the superposition so that there is no region of wholly-contained fields. If $\alpha^2 > \lambda_m^2$ for some integer m but not equal to any one of the eigenvalues $\alpha_{m', n'}^2$, the field has periodic structures in the z -direction along the dominant flux H_0 , corresponding to the solutions (27). In the range $\alpha^2 < \lambda_1^2$, where $\lambda_1 R_0$ is the first zero of Bessel function J_1 , all the terms of the anchored field (30) are not periodic in z and

do exponentially decay away from the boundaries $z = \pm L$, leaving a z -invariant field in the main length of the cylindrical domain, i.e., for L sufficiently large, which is an example of the general feature of analytic force-free fields discovered by the Parker magnetostatic $\epsilon = d/L \ll 1$ expansion.

As noted for the Woltjer force-free fields, the nonexistence of an anchored field with α being one of the Woltjer eigenvalues arises from the assumption of a continuous field. Allowing for the presence of TDs, a larger continuum of force-free states is available, and the restriction on the constant α by the Sturm-Liouville theory becomes irrelevant physically.

5. The z -invariant fields. Represent these constant- α fields as

$$\mathbf{B} = \frac{1}{R} \frac{\partial A}{\partial \varphi} \hat{\mathbf{R}} - \frac{\partial A}{\partial R} \hat{\boldsymbol{\varphi}} + Q \hat{\mathbf{z}}, \quad (31)$$

governed by

$$Q(R, \varphi) = Q(A) = \alpha A, \quad (32)$$

$$\nabla^2 A + \alpha^2 A = 0, \quad (33)$$

possessing the separable solutions

$$A = J_N(\alpha R) \cos(N\varphi), \quad (34)$$

$$Q = \alpha J_N(\alpha R) \cos(N\varphi), \quad (35)$$

where N is an integer.

Denote by $\xi_{N,M}$ the Bessel zeros, i.e., $J_N(\xi_{N,M}) = 0$, where $N = 1, 2, 3, \dots$ and $M = 1, 2, 3, \dots$ which define the eigenvalues $\alpha^2 = \alpha_{N,M}^2 = \xi_{N,M}^2/R_0^2$. Each eigenvalue $\alpha_{N,M}^2$ gives a twisted force-free field $\mathbf{B} = \mathbf{B}_{N,M}(R, \varphi)$ satisfying the *inhomogeneous* boundary conditions,

$$\text{at } R = R_0, \quad B_R = 0, \quad (36)$$

$$\text{at } z = \pm L, \quad B_z = \alpha_{N,M} J_N(\alpha_{N,M} R) \cos(N\varphi). \quad (37)$$

If $\alpha^2 \neq \alpha_{N,M}^2 = \xi_{N,M}^2/R_0^2$ for all (N, m) , the anchored force-free fields satisfying boundary conditions (36) and (37) must vary with all three coordinates (R, φ, z) , a formidable BVP to be treated in Conversation VII. For the special cases of $\alpha^2 = \alpha_{N,M}^2 = \xi_{N,M}^2/R_0^2$, the anchored force-free field $\mathbf{B}_{N,M}(R, \varphi)$ is z -invariant, in contrast to the (φ -invariant) axisymmetric anchored field $\mathbf{B}(R, z; \alpha)$ given by equation (30). The constant- α force-free equation (2) is linear in \mathbf{B} . Therefore, a linear superposition of $\mathbf{B}_{N,M}(R, \varphi)$ and $\mathbf{B}(R, z; \alpha)$ of the same constant $\alpha^2 = \alpha_{N,M}^2 = \xi_{N,M}^2/R_0^2$ gives a force-free field that does vary with all three coordinates (R, φ, z) .

Noteworthy is the special case of $N = 0$ in solution (34) and (35) which is independent of φ and satisfying boundary conditions (36) and (37) for all values of α . The superposition of this φ -independent $N = 0$ solution and a $N \neq 0$ solution with the same constant $\alpha = \alpha_{N,M}$,

$$A_s = J_0(\alpha_{N,M}R) + \gamma J_N(\alpha_{N,M}R) \cos(N\varphi), \quad (38)$$

with a free amplitude γ is a true anchored-field solution. The field \mathbf{B}_s constructed by this special superposition is presented in Figures 5 and 6.

6. A Medley of Explicit Solutions. The solutions are described in groups illustrated by the 6 figures below.

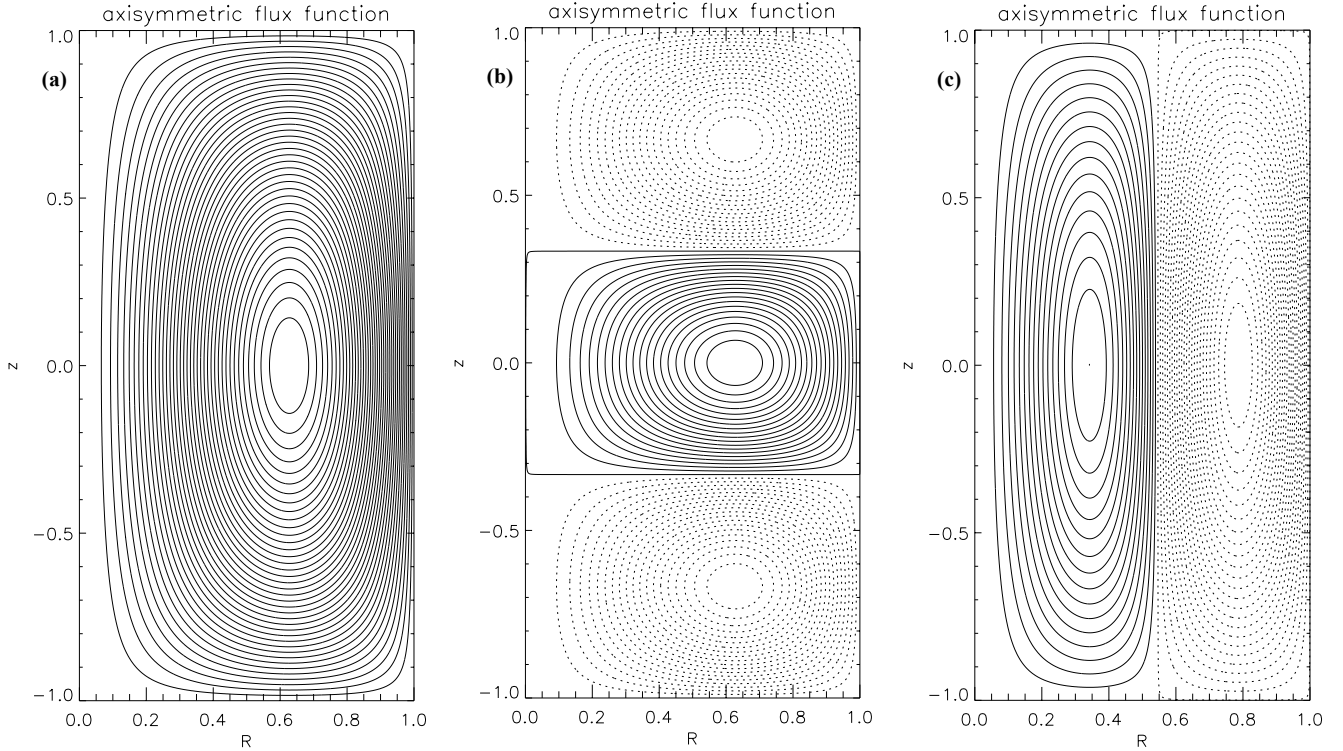


Figure 1. Axisymmetric Woltjer fields in domain $0 < R < R_0 = 1$, $|z| < L = 1$; (18) and (19), displaying the poloidal field-lines as contours of constant flux function $H = H_{m,n}(R, z) = RJ_1(\lambda_m R) \cos(\kappa_n z)$, with $\alpha^2 = \alpha_{m,n}^2 = \lambda_m R_0^2 + \kappa_n^2$: (a) $m = 1, n = 0, \xi_1 = \lambda_1 R_0 = 3.83171, \kappa_0 = \pi/2 = 1.570, \alpha_{1,0} = 4.140$, (b) $m = 1, n = 1, \xi_1 = 3.83171, \kappa_1 = 3\pi/2 = 4.712, \alpha_{1,0} = 6.073$, (c) $m = 2, n = 0, \xi_2 = 7.01559, \kappa_0 = \pi/2 = 1.570, \alpha_{1,0} = 7.189$. In Figures 1-4, solid and broken contours indicate positive and negative $H = H_{m,n}(R, z)$, respectively, with $B_\varphi = \alpha_{m,n} H_{m,n}/R$ along the H -contours.

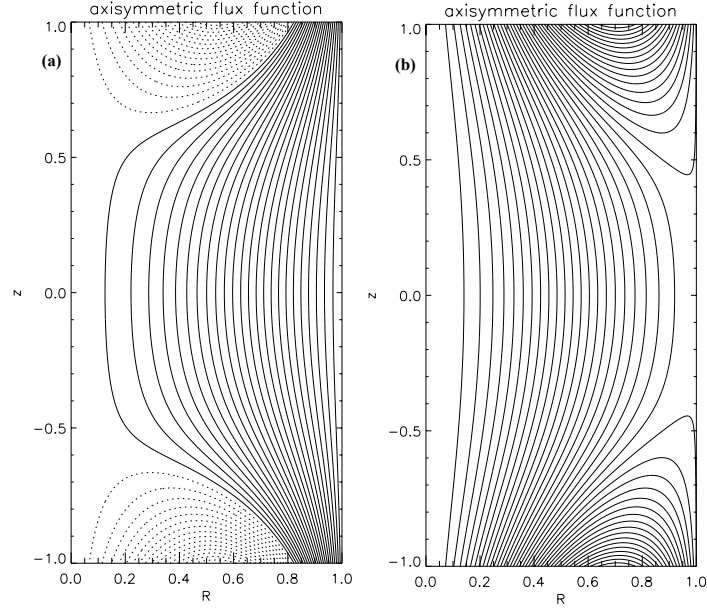


Figure 2. Axisymmetric $\alpha = 2.0$ anchored fields, in $R_0 = 1.0, L = 1.0$ domain; see equation (30). Contours of constant flux function $H = RJ_1(\alpha R) + \text{amplitude} \times RJ_1(\lambda_1 R) \frac{\cosh kz}{\cosh kL}$, $\xi_1 = \lambda_1 R_0 = 3.83171, k = \sqrt{\lambda_1^2 - \alpha^2} = 3.2683$: (a) amplitude = -1.0, (b) amplitude = 1.0.

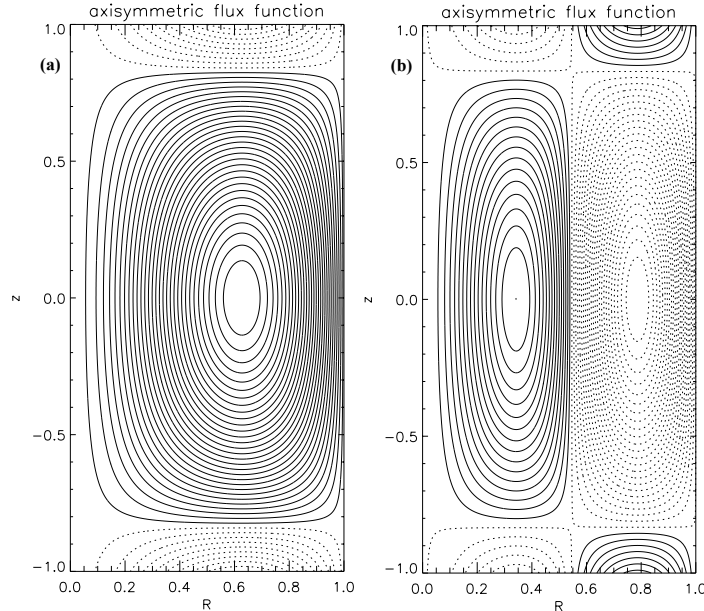


Figure 3. Axisymmetric anchored fields, in $R_0 = 1.0, L = 1.0$ domain; see equation (27). Contours of constant flux function $H = RJ_1(\lambda_m R) \frac{\cos \kappa z}{\cos \kappa L}$: (a) $\xi_1 = \lambda_1 R_0 = 3.83171, \kappa = (1 \times 0.1 + 1)0.5\pi = 1.885, \alpha = \sqrt{\lambda_1^2 + \kappa^2} = 4.270$, (b) $\xi_1 = \lambda_2 R_0 = 7.01559, \kappa = 1.885, \alpha = \sqrt{\lambda_1^2 + \kappa^2} = 7.264$.

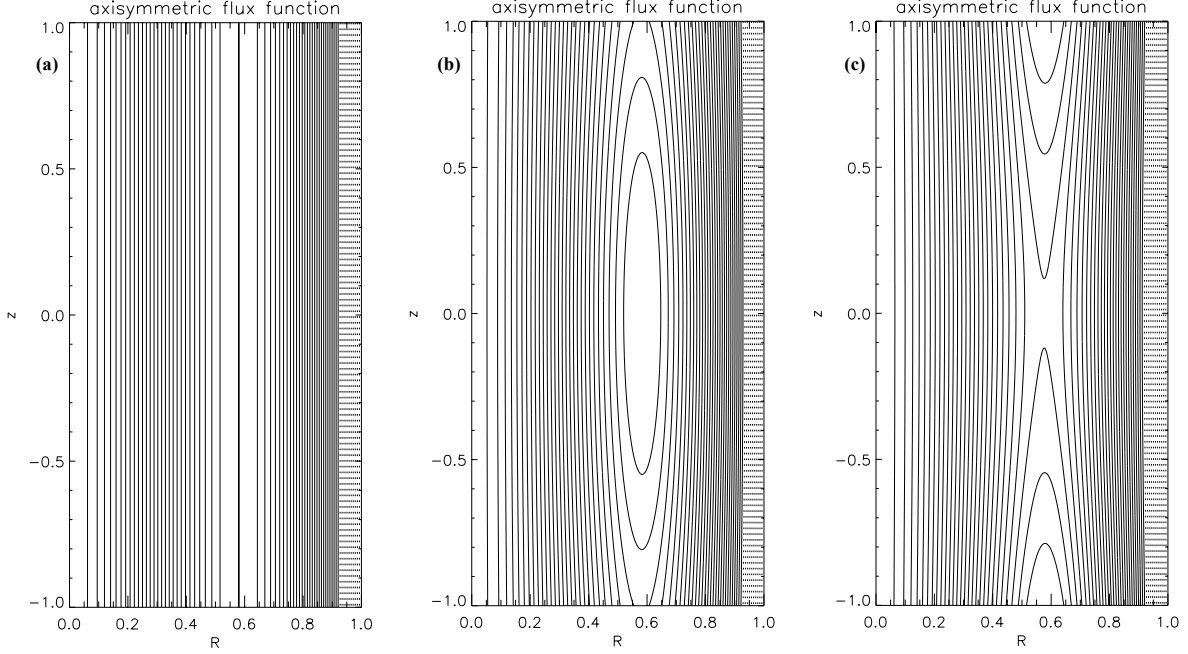


Figure 4. Axisymmetric anchored fields, in $R_0 = 1.0, L = 1.0$ domain; case of $\alpha = \alpha_{1,0} = \sqrt{\lambda_1^2 + \kappa_0^2} = 4,140$, a Woltjer eigenvalue with $\lambda_1 R_0 = 3.83171, \kappa_0 = \pi/2 = 1.570$ associated with "wholly-contained" flux function $H_{1,0}(R, z) = RJ_1(\lambda_1 R) \cos(\kappa_0 z)$, see Figure 1(a). The anchored solution (27) is not admissible with $\alpha = \alpha_{1,0} = 4,140, \kappa = \kappa_0 = \pi/2$ for which $\cos \kappa = 0$. Anchored field solutions do exist with $\alpha = \alpha_{1,0}$, the simplest being $H_1(R) = RJ_1(\alpha_{1,0} R)$ describing the z -invariant field $\mathbf{B} = \frac{1}{R} \frac{dH_1}{dR} \hat{\mathbf{z}} = \alpha_{1,0} J_0(\alpha_{1,0} R) \hat{\mathbf{z}}$, field-lines of constant H_1 shown in (a). The linear superposition $H_1 + \gamma H_{1,0}(R, z)$ is a $\alpha = \alpha_{1,0}$ force-free field for any constant amplitude γ , all these fields sharing the same normal flux distribution at $z = \pm L$, the cases $\gamma = 0.1, -0.1$ shown in (b) & (c), respectively.

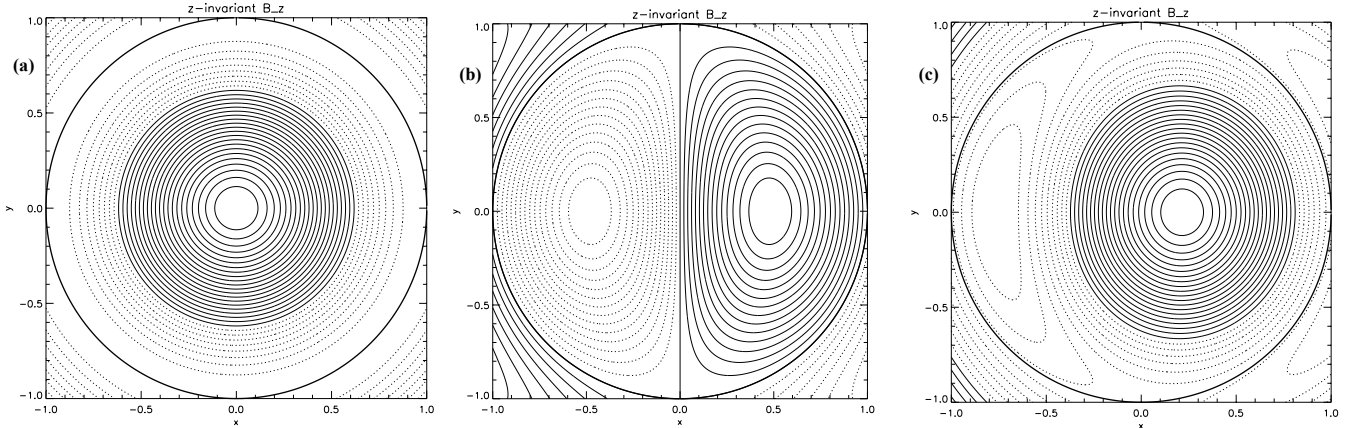
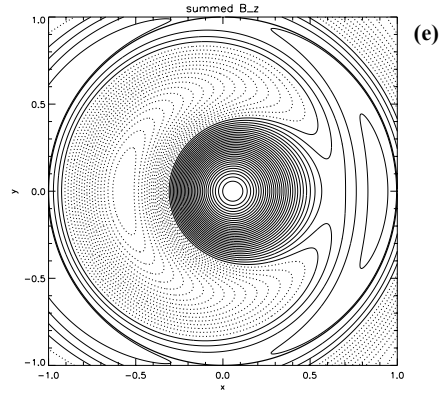
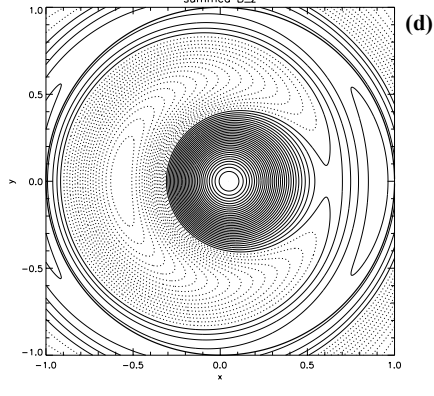
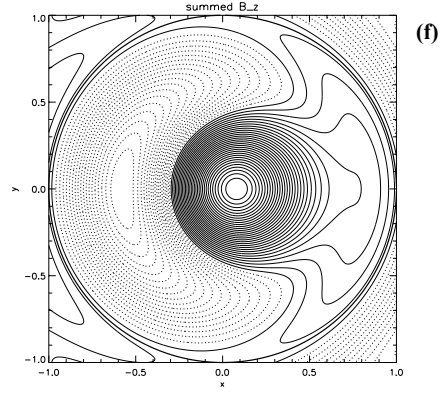
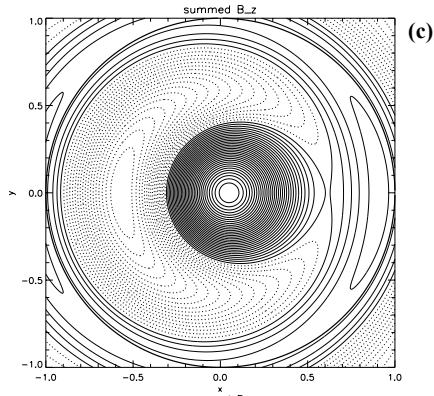
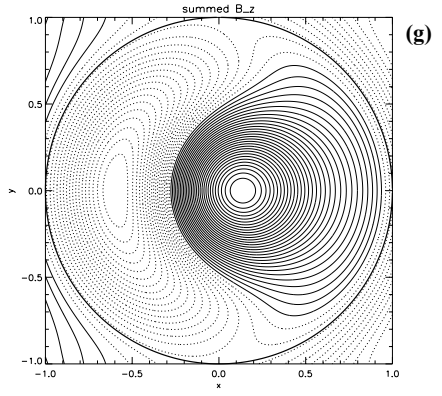
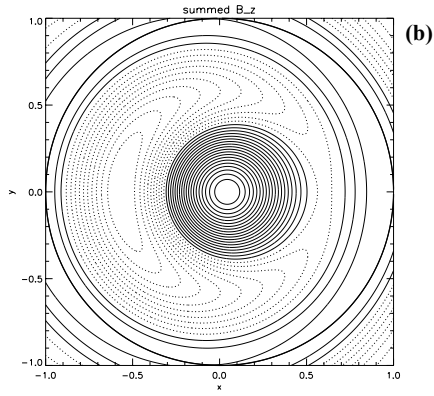
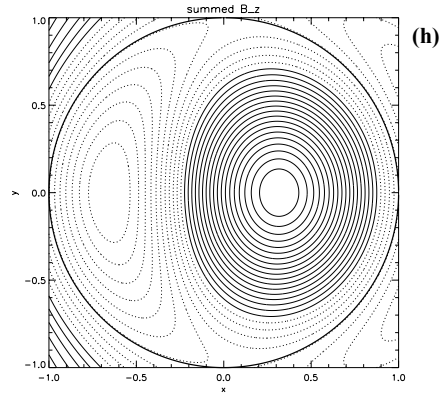
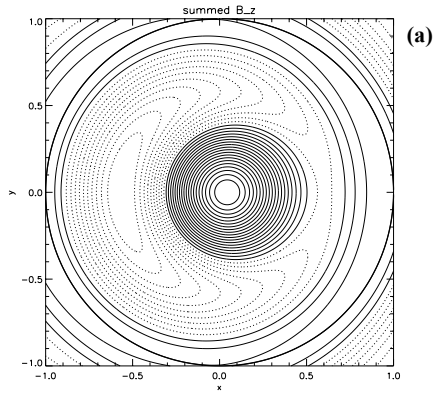


Figure 5. The z -invariant field \mathbf{B}_s , generated by the superposition (38) with amplitude $\gamma = 1$.

(a) The A -contours of the φ -independent first term of the superposition, $J_0(\alpha_{0,0}R)$, are concentric circles. By solution (35), the A -contours may be interpreted as contours of B_z on the $R - \varphi$ plane. Solid and broken contour lines indicate positive and negative values of A , (a) showing a $B_z > 0$ core surrounded by an external $B_z < 0$ layer. (b) The A -contours of the φ -dependent second term of the superposition, $J_1(\alpha_{1,1}R)\cos\varphi$, is composed of a symmetrically placed pair of flux ropes of opposite-signed B_z . (c) The superposition $A = J_0(\alpha_{1,1}R) + J_1(\alpha_{1,1}R)\cos\varphi$ displays the combined field comprising a $B_z > 0$ rope embedded in a complex $B_z < 0$ external layer.

Figure 6. The superposition of the z -invariant \mathbf{B}_s in Figure 5 with the axisymmetric, φ -invariant $\mathbf{B}(R, z; \alpha)$ with $\alpha = \alpha_{1,1}, m = 2, k = \sqrt{\lambda_2^2 - \alpha_{1,1}^2}$ in solution (28). The field $\mathbf{B}(R, z; \alpha)$ decays with z away from both boundaries $z = \pm L$ to leave the z -invariant \mathbf{B}_s in the main length of the cylindrical domain. This property is shown for the combined field $\mathbf{B}(R, z; \alpha) + \mathbf{B}_s$ in the $R < R_0 = 1, |z| < L = 1$ domain, by displaying the contours of B_z of the combined field at a series of select constant- z planes labelled (a) - (h), from subfigure (a) at $z = -L$ to successive subfigures at increasing z towards the mid-point $z = 0$ of the cylindrical domain. Solid and broken contour lines indicate positive and negative B_z , respectively. At (a) on $z = -L$, a core $B_z > 0$ rope is seen surrounded by a complex $B_z < 0$ layer that abuts another, outer, layer with $B_z > 0$ running along the boundary $R = R_0$. We may infer that the local 3D flux emerges out of the outer $B_z > 0$ layer to return low to anchor at $z = -L$, whereas a core field of both signs in the central region do rise as twisted flux ropes clear to reach the opposite boundary $z = L$ as suggested by the contour map (h). The B_z -contour maps (a) - (h) have been selected at constant- z planes spaced apart to best reveal the static organization of B_z from within the boundary layers at $z \pm L$ into the simpler rope structure in (h).



Conversations with Eugene N. Parker¹

VI. Various Further Thoughts

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May 30, 2020

This Conversation VI summarizes further thoughts on several magnetostatic properties discussed in our May 29, 2020 phone call, referencing the Book *Spontaneous Current Sheets in Magnetic fields*, Parker (1994). Conversation V on a complete analysis of the constant- α fields is still being edited for smooth reading.

1. Magnetostatic Theorem is pointed out as a complementary approach in relation to traditional MHD studies of resistive non-equilibrium; see the Book's 10.3.6 The Complementary Approach & 10.3.7 The third Dimension. From its first presentation in 1972, this theorem makes a radical statement that, stripped of the rich complexity of the time-dependent MHD processes, force equilibrium has a direct physical relationship with the topology of the magnetic field preserved under the frozen-in condition of a perfect fluid conductor, namely, that *most* field topologies in 3D fluid systems are incompatible with an everywhere continuous equilibrium state. The inevitable presence of TDs in *most* equilibrium states naturally implies that the TDs must generally arise in time-dependent dynamics. Thus, for $R_m \gg 1$ MHD, this relationship is at the root of tearing-mode and related resistive instabilities studied over the years. More than a complementary approach, the Theorem has identified a fundamental property of the perfectly conducting fluid not previously known or conceptualized as such. It makes specific what is meant by field topology, magnetic TDs, and the perfect conductor as a singular limit of $R_m \rightarrow \infty$. The above view of the Theorem is an instructive way to introduce the Theorem to the Crafoord Symposium.

2. A Good set of statements on why TDs are inevitable in force-free fields.

(a) The magnetic field lines are the characteristics for α , a non-trivial nonlinear complication to the equilibrium equation, except in the case of a spatially uniform α . Note that a discontinuity of α across a flux surface does necessarily imply a TD. Such a discontinuity in α by the equation $\nabla \times \mathbf{B} = \alpha \mathbf{B}$ implies that the gradients of \mathbf{B} are discontinuous without necessarily requiring \mathbf{B} to be discontinuous.

(b) Flux expulsion from between two or more initially spatially apart, distinct flux systems

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in response to Lorentz and other body forces, an essentially 3D effect. When limited to 2D, like in the Cartesian plane, multiple fluxes are possible only by the presence of magnetic neutral points and their associated separatrix lines, whereas neutral points are not essential for the process in a 3D system. By fully 3D forcing, any flux surface may be punched with a hole where a TD forms, as described by the Optical Analogy treated in Chapter 7.

(c) Flux expulsion among closely packed flux tubes of continuous fields has the uncontrollable propensity for lower energy states of Y-type, three-tube contacts versus X-type, four-tube contacts, the latter arising only under the imposition of an everywhere spatially continuous equilibrium field.

(d) The role of the rigid boundary is essential in TD formation. It is the anchoring of a field by the rigid, perfectly conducting wall that holds a field to its invariant field topology under the induction equation, similarly for the topology of the field wholly contained in the fluid domain. TDs form in the field that would otherwise have been avoided if a local field topology forcing a TD could propagated away permanently as nonlinear Alfvén waves out of the locality. For the anchored field, a TD forming in the fluid interior would extend all the way along the field to the rigid boundary, producing a point infinity in field intensity, integrable for finite energy, at the boundary. This effect was overlooked in Fig. 8.11, page 236, *Spontaneous Current Sheet*, also see Conversation IV. Useful to see explicit global solutions of force-free fields, with and without TDS, illustrating the connections of their boundary layers to the interior equilibrium structures, of interest in particular to the $L \rightarrow \infty$ limit of the small-parameter expansion for an analytical force-free fields, discussed in the 1972 Magnetostatic Theorem paper.

3. The constant- α force-free fields of Conversation V will be found to be interesting in their own right as explicit examples of continuous equilibrium. The uniform α removes the formidable nonlinearity of the field lines being the characteristics for a non-uniform α , but the former case with its linearly superposable equilibrium fields presents its own peculiar class of existence and non-existence of solutions and uniqueness and non-uniqueness of solutions to boundary value problems. Acquaintance with these special linear fields provides insight for how TDs are inevitable in the general nonlinear fields

A topological property first identified in the 1972 Magnetostatic Theorem is that the phase-space of nearby *continuous* equilibrium states, represented by the $\epsilon \ll 1$ expansion, around the uniform field is sparsely distributed among the equilibrium states containing inevitable TDs, sparsity in the sense of that characterized by the distribution of rational fractions among the irrational numbers in the interval from zero to 1. One must similarly imagine that the phase space of nearby equilibrium states of different topologies around each of the constant- α force-free field are *mostly* studded with TDs as opposed to the sparse other,

everywhere continuous equilibrium fields.

Explicit constant- α fields in the full 3D varieties are interesting for investigating what equilibrium states are available to each constant- α field when subject to cleverly designed foot-point displacements at the boundary. This investigation may prove even more interesting than the case treated in the 1972 paper for the rather unsophisticated uniform field anchored by the two-plate boundary.

Conversation V addresses a non-trivial complication presented by the third spatial dimension in force-free equilibrium. For the cylindrical domain, the 3D varying force-free field has the Chandrasekhar poloidal-toroidal $\Phi - \Psi$ representation as the obvious appropriate description, but the 3D cylindrical boundary conditions are manifestly over-determining on $\Phi - \Psi$ for a constant- α field. A re-derivation of the governing constant- α force-free equation for $\Phi - \Psi$ has caught a hidden freedom for removing the over-determination, with the surprise appearance of the Cauchy-Riemann equations of complex analysis. Here is an example of how a fully 3D equilibrium field is a mathematically distinct object from its 2D counterpart, even in the realm of spatially continuous fields.

Conversations with Eugene N. Parker¹

VII. The General 3D Constant- α Force-Free Fields

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June 15, 2020

We pick up the discussion on page 308 of Parker's Book, on the complexity of the magnetostatic equilibrium in the presence of variation in the third spatial dimension.

In an intuitive sense, the Parker Theorem captures a certain impossibility for the anisotropic Lorentz force of continuous magnetic field \mathbf{B} to balance itself or against pressure gradient in the perfect fluid conductor unless suitably restrained by some geometric symmetry imposed on the system. The equilibrium equation

$$\frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B} - \nabla p = 0, \quad (1)$$

requires the field to be restricted to lie on global flux surfaces of constant pressure p , a prerequisite readily met, for example, if the system is describable under an ignorable coordinate. This means that a 3D non-equilibrium field must possess such global flux surfaces to begin with that would then allow for a continuous deformation under the induction equation to establish the isobaric fluid surfaces for equilibrium.

This topological constraint on the field is prohibitively strong, most obviously seen in the infinitely long, volume-filling field lines generally occurring in the arbitrary 3D field. The field-lines in Cartesian coordinates defined by

$$\frac{dx}{B_x} = \frac{dy}{B_y} = \frac{dz}{B_z}, \quad (2)$$

requires global constraints on a 3D $\mathbf{B}(x, y, z)$ for the flux surfaces to exist as global geometric surface. The mathematics of this interesting issue takes us to the complex mathematics of integrability of ODEs in spatial dimensions greater than 2 and chaos theory. But the intuitive ideas are clear for our physical purposes here, that most 3D fields do not meet the prerequisites for 3D flux surfaces to exist. As Parker has said to BC once, a field line has zero cross section and carries no flux. Closed field lines are exceptional, for it is impossible for a solenoidal line without strong constraints imposed on it to hit itself in the rear.

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Anchored field lines may avoid infinitely long field lines, but not always the case, in quite simple examples to be taken up in a future Conversation. We need to get through Conversations V, VI and VII, to build an intuitive basis for linking concepts to explicit solutions.

Moreover, even if global flux surfaces exist in an equilibrium field with an ignorable coordinate, a breakage of the frozen-in condition by some resistive effects would "destroy" the global form of the original flux surfaces, whereupon the woes of absence of global flux surfaces to define isobaric surface, once incurred, put the field into an irrecoverable state of non-equilibrium.

In the case of a negligible pressure, the force-free fields of vanishing Lorentz force are governed by

$$\nabla \times \mathbf{B} = \alpha \mathbf{B}, \quad (3)$$

$$\mathbf{B} \cdot \nabla \alpha = 0, \quad (4)$$

where we see that equilibrium still requires existence of global flux surfaces as surfaces of constant α . This prerequisite for equilibrium can be stated more strongly by noting that the equilibrium equations are a mix of elliptic and hyperbolic PDEs, with the field lines as the characteristic; equation (4), along which α is propagated as a constant, see 3.4 Characteristics & Discontinuities, page 61 of Book. This nonlinear complication of the characteristics is removed in the simplest force-free fields, namely, those with a spatially uniform α . There is then no foliation of surfaces of different constant α -values on them to deal with. Still, in the absence of an ignorable coordinate, a 3D fully varying constant- α fields are physically and mathematically distinct from those possessing an ignorable coordinate treated in Conversation V.

Here 2 things are done, the first to properly formulate the BVP for a 3D field in the cylindrical domain of Conversation V, pointing out a flaw in many previous journal publications discovered and resolved by Low & Fang (2014). We will give a full solution of the BVP that looked formidably intractable to Low & Fang in 2014.

1. The 3D constant- α fields. Consider a 3D field in the cylindrical domain $R < R_0, -L < z < L$ subject to boundary conditions

$$R = R_0, \quad B_R = 0; \quad z = \pm L, \quad B_z = F(R, \varphi), \quad (5)$$

in cylindrical coordinates. The field enters and leaves the domain at $z = \pm L$ with the same normal flux distribution, tangential along the cylindrical side.

Represent the field by the Chandrasekhar poloidal-toroidal flux functions:

$$\mathbf{B} = \mathbf{B}_\Psi + \mathbf{B}_\Phi, \quad (6)$$

$$\mathbf{B}_\Psi = \nabla \times (\nabla \times [\Psi \hat{\mathbf{z}}]) = \nabla \frac{\partial \Psi}{\partial z} - \nabla^2 \Psi \hat{\mathbf{z}} = \nabla_\perp \frac{\partial \Psi}{\partial z} - \nabla_\perp^2 \Psi \hat{\mathbf{z}}, \quad (7)$$

$$\mathbf{B}_\Phi = \nabla \times [\Phi \hat{\mathbf{z}}] = \frac{1}{R} \frac{\partial \Phi}{\partial \varphi} \hat{\mathbf{R}} - \frac{\partial \Phi}{\partial R} \hat{\varphi}, \quad (8)$$

$$\mathbf{B} = \left(\frac{\partial^2 \Psi}{\partial R \partial z} + \frac{1}{R} \frac{\partial \Phi}{\partial \varphi} \right) \hat{\mathbf{R}} + \left(\frac{1}{R} \frac{\partial \Psi}{\partial \varphi \partial z} - \frac{\partial \Phi}{\partial R} \right) \hat{\varphi} - \nabla_\perp^2 \Psi \hat{\mathbf{z}}, \quad (9)$$

$$\nabla \times \mathbf{B} = \left(\frac{\partial^2 \Phi}{\partial R \partial z} - \frac{1}{R} \frac{\partial}{\partial \varphi} \nabla^2 \Psi \right) \hat{\mathbf{R}} + \left(\frac{1}{R} \frac{\partial \Phi}{\partial \varphi \partial z} + \frac{\partial}{\partial R} \nabla^2 \Psi \right) \hat{\varphi} - \nabla_\perp^2 \Phi \hat{\mathbf{z}}, \quad (10)$$

$$\nabla_\perp \equiv \hat{\mathbf{R}} \frac{\partial}{\partial R} + \frac{1}{R} \hat{\varphi} \frac{\partial}{\partial \varphi}; \quad \nabla_\perp^2 \equiv \frac{1}{R} \frac{\partial}{\partial R} \left(R \frac{\partial}{\partial R} \right) + \frac{1}{R^2} \frac{\partial^2}{\partial \varphi^2}. \quad (11)$$

Force-free equation (4) is trivially satisfied with α constant. Substitution into force-free equation (3), we obtain

$$\frac{\partial^2 \Phi}{\partial R \partial z} - \frac{1}{R} \frac{\partial}{\partial \varphi} \nabla^2 \Psi = \alpha \left(\frac{\partial^2 \Psi}{\partial R \partial z} + \frac{1}{R} \frac{\partial \Phi}{\partial \varphi} \right), \quad (12)$$

$$\frac{1}{R} \frac{\partial \Phi}{\partial \varphi \partial z} + \frac{\partial}{\partial R} \nabla^2 \Psi = \alpha \left(\frac{1}{R} \frac{\partial \Psi}{\partial \varphi \partial z} - \frac{\partial \Phi}{\partial R} \right), \quad (13)$$

$$\nabla_\perp^2 \Phi = \alpha \nabla_\perp^2 \Psi. \quad (14)$$

Rewrite equations (12) and (13) as

$$\frac{\partial}{\partial R} \frac{\partial}{\partial z} (\Phi - \alpha \Psi) = \frac{1}{R} \frac{\partial}{\partial \varphi} (\nabla^2 \Psi + \alpha \Phi), \quad (15)$$

$$\frac{1}{R} \frac{\partial}{\partial \varphi} \frac{\partial}{\partial z} (\Phi - \alpha \Psi) = -\frac{\partial}{\partial R} (\nabla^2 \Psi + \alpha \Phi), \quad (16)$$

which are, for the first time recognized by Low & Fang (2014), the Cauchy-Riemann equations of complex analysis (Ahlfors 1953).

Consider $W(\omega, z) = \xi + i\zeta$, a function of complex variable $\omega = x + iy$ and real z . The real and imaginary parts of W satisfy the Cauchy-Riemann equations

$$\frac{\partial \xi}{\partial x} = \frac{\partial \zeta}{\partial y}, \quad (17)$$

$$\frac{\partial \xi}{\partial y} = -\frac{\partial \zeta}{\partial x}, \quad (18)$$

that take the form

$$\frac{\partial \xi}{\partial R} = \frac{1}{R} \frac{\partial \zeta}{\partial \varphi}, \quad (19)$$

$$\frac{1}{R} \frac{\partial \xi}{\partial \varphi} = -\frac{\partial \zeta}{\partial R}, \quad (20)$$

under coordinate transformation $(x, y) = (R \cos \varphi, R \sin \varphi)$. By inspection of the pair of equations (15) and (16) against equations (19) and (20) we deduce that the pair has the integral

$$\Phi - \alpha\Psi = \xi, \quad (21)$$

$$\nabla^2\Psi + \alpha\Phi = \frac{\partial\zeta}{\partial z}, \quad (22)$$

where $W(\omega, z) = \xi + +i\zeta$ is a free function. Eliminating Φ , we obtain the PDE for Ψ ,

$$\nabla^2\Psi + \alpha^2\Psi = \frac{\partial\zeta}{\partial z} - \alpha\xi, \quad (23)$$

coupled to the 2D Laplace equations

$$\nabla_{\perp}^2\xi = 0, \quad \nabla_{\perp}^2\zeta = 0, \quad (24)$$

determining (ξ, ζ) as a pair of conjugate potential functions on planes of constant z . Force-free equation (14) is also satisfied by virtue of equation (21) and $\xi(R, \varphi, z)$ being a harmonic function of (R, φ) .

Boundary conditions (5) in the Chandrasekhar representation read:

$$R = R_0, \quad \frac{\partial^2\Psi}{\partial R\partial z} + \frac{1}{R} \frac{\partial\Phi}{\partial\varphi} = 0; \quad (25)$$

$$z = \pm L, \quad \nabla_{\perp}^2\Psi = -F(R, \varphi). \quad (26)$$

By equation (21), boundary condition (25) gives

$$R = R_0, \quad \frac{\partial^2\Psi}{\partial R\partial z} - \frac{1}{R} \frac{\partial}{\partial\varphi} (\alpha\Psi + \xi) = 0. \quad (27)$$

Boundary conditions (26) and (27) involving the second order derivatives of Ψ are not standard for the second order elliptic PDE (23) governing Ψ , requiring further physical consideration.

The Chandrasekhar decomposition (6) accounts for B_z everywhere in the cylindrical domain in terms of \mathbf{B}_{Ψ} alone, \mathbf{B}_{Φ} being a field with zero z -component. For the field \mathbf{B} to be tangential at the side boundary $R = R_0$, the radial components of \mathbf{B}_{Ψ} and \mathbf{B}_{Φ} must cancel, represented by the two respective terms on the left side of equation (27). There is mathematical freedom to set these two radial components to zero separately. To that end, we first transform boundary condition (26) into a Dirichlet boundary condition on Ψ at $z = \pm L$.

The net magnetic flux

$$\begin{aligned}
 F_0 &= \int_0^{R_0} \int_0^{2\pi} [B_z]_{z=\pm L} d\varphi R dR \\
 &= \int_0^{R_0} \int_0^{2\pi} F(R, \varphi) d\varphi R dR.
 \end{aligned} \tag{28}$$

is channeled as a constant along the length of the domain. Using equation (7) and an application of Stokes Theorem, we have

$$\begin{aligned}
 F_0 &= \int_0^{R_0} \int_0^{2\pi} B_z d\varphi R dR \\
 &= \int_0^{R_0} \int_0^{2\pi} \mathbf{B}_\Psi \cdot d\varphi R dR \hat{\mathbf{z}} \\
 &= \int_0^{R_0} \int_0^{2\pi} \nabla \times (\nabla \times [\Psi \hat{\mathbf{z}}]) \cdot d\varphi R dR \hat{\mathbf{z}} \\
 &= \int_0^{2\pi} \left[\frac{\partial \Psi}{\partial R} \right]_{R=R_0} R d\varphi,
 \end{aligned} \tag{29}$$

independent of the fixed- z where these integrals are evaluated. That is, the boundary values of $\frac{\partial \Psi}{\partial R}$ along each circle $R = R_0$ at fixed z is restricted to have an average $\frac{F_0}{2\pi R_0}$ independent of z .

Let us then impose and justify later the boundary conditions on Ψ governed by PDE (23) :

$$\text{at } z = \pm L, \quad \Psi = \Psi_L(R, \varphi), \tag{30}$$

$$\text{at } R = R_0, \quad \frac{\partial \Psi}{\partial R} = \frac{F_0}{2\pi R_0}, \tag{31}$$

where $\Psi_L(R, \varphi)$ is the unique solution to the potential BVP on $z = \pm L$,

$$\nabla_\perp^2 \Psi_L = 0 ; \quad \text{at } R = R_0, \quad \frac{\partial \Psi_L}{\partial R} = \frac{F_0}{2\pi R_0}. \tag{32}$$

Neumann condition (31) implies that the second-order mixed derivative of Ψ is zero in boundary condition (27), corresponding to the vanishing of the radial component of \mathbf{B}_Ψ at $R = R_0$. The remaining term in boundary condition (27) is the radial component of \mathbf{B}_Φ that must also vanish, leading to the Dirichlet boundary condition on ξ ,

$$\text{at } R = R_0, \quad \xi = -\alpha \Psi. \tag{33}$$

Cauchy-Riemann equation (20) then poses the Neumann boundary condition

$$\text{at } R = R_0, \quad \frac{\partial \zeta}{\partial R} = \frac{\alpha}{R_0} \frac{\partial}{\partial \varphi} \Psi(R_0, \varphi, z), \quad (34)$$

on the harmonic conjugate ζ .

The important point should be emphasized that the boundary value of Ψ at $R = R_0$ is an unknown in the problem, in terms of which (ξ, ζ) as unique solutions to Laplace equations (24) have the following Green's function representations:

$$\xi(R, \varphi, z) = -\frac{\alpha}{\pi} \int_0^{2\pi} \Psi(R_0, \varphi', z) \frac{R_0 - R \cos(\varphi - \varphi')}{R^2 + R_0^2 - 2RR_0 \cos(\varphi - \varphi')} R_0 d\varphi', \quad (35)$$

$$\zeta(R, \varphi, z) = \frac{\alpha}{\pi} \int_0^{2\pi} \frac{\partial}{\partial \varphi'} \Psi(R_0, \varphi', z) \frac{R_0 - R \sin(\varphi - \varphi')}{R^2 + R_0^2 - 2RR_0 \cos(\varphi - \varphi')} R_0 d\varphi'. \quad (36)$$

The governing PDE (23) thus takes the form of an integro-PDE,

$$\begin{aligned} \nabla^2 \Psi + \alpha^2 \Psi &= \frac{\alpha}{\pi} \int_0^{2\pi} \frac{\partial^2}{\partial \varphi' \partial z} \Psi(R_0, \varphi', z) \frac{R_0 - R \sin(\varphi - \varphi')}{R^2 + R_0^2 - 2RR_0 \cos(\varphi - \varphi')} R_0 d\varphi' \\ &+ \frac{\alpha^2}{\pi} \int_0^{2\pi} \Psi(R_0, \varphi', z) \frac{R_0 - R \cos(\varphi - \varphi')}{R^2 + R_0^2 - 2RR_0 \cos(\varphi - \varphi')} R_0 d\varphi', \end{aligned} \quad (37)$$

to determine the single variable Ψ subject to boundary conditions (30) and (31), noting that the boundary values of Ψ and its tangential derivative at $R = R_0$ are not prescribed but are to be solved for self-consistently.

Four concluding remarks:

(i) The Chandrasekhar decomposition is not unique for a given a field \mathbf{B} . \mathbf{B}_Ψ has the same z -component as \mathbf{B} , fixed by $\nabla_\perp^2 \Psi = -B_z$, whereas the R - φ components of \mathbf{B}_Ψ are defined by Ψ depending on the (free to specify) boundary condition for Ψ on $R = R_0$ at fixed z . Once \mathbf{B}_Ψ is rendered definite, $\mathbf{B}_\Phi = \mathbf{B} - \mathbf{B}_\Psi$ also definite. In our cylindrical BVP problem on a field \mathbf{B} tangential along $R = R_0$, we are free to set \mathbf{B}_Ψ tangential so that \mathbf{B}_Φ is also tangential along that boundary, respectively expressed by boundary conditions (31) and (33). With the conjugate harmonic pair (ξ, ζ) as auxiliary functions accounting for the R - φ field components in the Chandrasekhar decomposition, no generality is lost in posing boundary condition (31) in compliance with the conservation of the net flux F_0 along the length of the cylindrical domain.

(ii). The Low & Fang (2014) derivation is novel in their recognition of equations (15) and (16) as Cauchy-Riemann equations, introducing the conjugate pair (ξ, ζ) essential for deriving the

integro-PDE (37). This fundamental step was missed in previous works in the journals by jumping unjustifiably from equations (15) and (16) to their particular solution

$$\Phi - \alpha\Psi = 0, \tag{38}$$

$$\nabla^2\Psi + \alpha\Phi = 0, \tag{39}$$

from which the Helmholtz PDE follows,

$$\nabla^2\Psi + \alpha^2\Psi = 0, \tag{40}$$

for which boundary condition (27) takes the form

$$R = R_0, \quad \frac{\partial^2\Psi}{\partial R\partial z} - \frac{\alpha}{R} \frac{\partial\Psi}{\partial\varphi} = 0. \tag{41}$$

There were only numerical attempts at 3D solutions for the BVP (40) and (41), iterating numerical solutions of the second order elliptic PDE to forcefully comply with a boundary condition containing a second order derivative, without an awareness of the inherent flaw of the problem.

(iii) The complication associated with the harmonic pair (ξ, ζ) for 3D fields disappears if either z or of φ is an ignorable, leading to the z -invariant and φ invariant sets of force-free field treated in Conversation V. We are reminded of such symmetric fields being distinct in a fundamental way from the general 3D fields without symmetry; page 308 on "the third dimension" of Parker's Book.

2. A family of general solutions, discovered in the course of these Conversations.

The reduction of the BVP to the integro-PDE (37) for the single unknown Ψ subject to boundary conditions (30) and (31) illuminates the basic structure of the problem. The following Fourier-series approach to the problem avoids the formidable governing integro-PDE in the case of zero net flux F_0 along the length of the cylindrical domain, replacing boundary condition (31) with the homogeneous condition

$$\text{at } R = R_0, \quad \frac{\partial\Psi}{\partial R} = 0. \tag{42}$$

The case of $F_0 \neq 0$ needs a separate treatment, not yet figured out.

Consider the Fourier series representation:

$$\Psi = \sum_{n=1}^{\infty} \exp(k_n z) [F_n(R) \cos n\varphi + G_n(R) \sin n\varphi], \tag{43}$$

$$\xi = \sum_{n=1}^{\infty} \exp(k_n z) [a_n R^n \cos n\varphi + b_n R^n \sin n\varphi], \tag{44}$$

$$\zeta = \sum_{n=1}^{\infty} \exp(k_n z) [a_n R^n \sin n\varphi - b_n R^n \cos n\varphi]. \quad (45)$$

Substitution into PDE (23) yields the ODEs by Fourier components:

$$\frac{d^2 F_n}{dR^2} + \frac{1}{R} \frac{dF_n}{dR} + \left(k_n^2 + \alpha^2 - \frac{n^2}{R^2} \right) F_n + (\alpha a_n + k_n b_n) R^n = 0, \quad (46)$$

$$\frac{d^2 G_n}{dR^2} + \frac{1}{R} \frac{dG_n}{dR} + \left(k_n^2 + \alpha^2 - \frac{n^2}{R^2} \right) G_n + (\alpha b_n - k_n a_n) R^n = 0, \quad (47)$$

which we recognize to be in the form of the ODE for the Lommel function

$$\frac{d^2 W_{n,n+1}}{ds^2} + \frac{1}{R} \frac{dW_{n,n+1}}{ds} + \left(k_n^2 + \alpha^2 - \frac{n^2}{s^2} \right) W_{n,n+1} + s^{n-1} = 0, \quad (48)$$

where $s^2 = (k_n^2 + \alpha^2) R_0^2$. Therefore the solutions to ODEs (46) and (47) are:

$$F_n = -\frac{\alpha a_n + k_n b_n}{(k_n^2 + \alpha^2)^{1+n/2}} W_{n,n+1}(s), \quad (49)$$

$$G_n = -\frac{\alpha b_n - k_n a_n}{(k_n^2 + \alpha^2)^{1+n/2}} W_{n,n+1}(s), \quad (50)$$

where we impose $W_{n,n+1} = 0$ at $s = 0$.

Boundary condition (42) requires $\frac{dW_{n,n+1}}{ds}(s) = 0$ at $R = R_0$. Denote by s_j the zeros of the first derivative of the Lommel function, for $j = 1, 2, 3, \dots$, which fixes $(k_n^2 + \alpha^2) R_0^2 = s_j^2$ for each zero. We also apply boundary condition (33):

$$a_n s_j^{n+2} - \alpha \frac{\alpha a_n + k_n b_n}{(k_n^2 + \alpha^2)^{1+n/2}} W_{n,n+1}(s_j) = 0, \quad (51)$$

$$b_n s_j^{n+2} - \alpha \frac{\alpha b_n - k_n a_n}{(k_n^2 + \alpha^2)^{1+n/2}} W_{n,n+1}(s_j) = 0, \quad (52)$$

which defines α as an eigenvalue as follows. For each zero $s = s_j$, $k_n^2 R_0^2 = s_j^2 - \alpha^2 R_0^2$ to express k_n in terms of α . Algebraic equations (51) and (52) are homogeneous in (a_n, b_n) to be solved for the ratio a_n/b_n and α simultaneously. The Fourier series can then be fitted to $\Psi_L(R, \varphi)$ prescribed under Dirichlet boundary condition (30) at $z = \pm L$, and the construction of the solution is then complete.

Conversations with Eugene N. Parker¹

VIII. Statement of the 1972 Parker Magnetostatic Theorem & Continuous and TD Solutions to the Two-Plate Problem

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Here we attempt at a physically direct, mathematically crisp statement of the 1972 Parker Magnetostatic Theorem, followed by a discussion of the two-plate problem with explicit force-free fields with and without TDs.

1. Statement of Theorem. Consider a continuous magnetic field \mathbf{B} in a perfect fluid conductor contained in a simply-connected domain V , each field line anchored at its pair of foot-points (FTs) on rigid, perfectly-conducting boundary ∂V where

$$F(\partial V) = \mathbf{B} \cdot \hat{\mathbf{n}} \neq 0 \quad (1)$$

is prescribed; $\hat{\mathbf{n}}$ the unit surface normal. The field evolves under the action of a continuous velocity \mathbf{v} by the induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}), \quad (2)$$

subject to $\mathbf{v} = 0$ on ∂V imposed by wall-rigidity and the tangential continuity of the electric field $\mathbf{E} = -\frac{1}{c}\mathbf{v} \times \mathbf{B}$ at ∂V . Taking the fluid to have negligible inertia, equilibrium is meant the force-free state described by

$$\nabla \times \mathbf{B} = \alpha \mathbf{B}, \quad (3)$$

$$\mathbf{B} \cdot \nabla \alpha = 0, \quad (4)$$

subject to boundary condition (1), under the tacit assumption that the field remains continuous in equilibrium. The Parker Theorem in 1972, nearly 50 years ago, showed that this assumption is generally not valid, thus permanently transforming our concept of magnetostatic equilibrium.

My statement of the Theorem had habitually started with the invariance of the field's topology T under the induction equation. By its textbook definition, T comprises all those

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properties of the magnetic flux surfaces, as geometric objects, that are invariant under *all* continuous deformations. Given a specific field \mathbf{B}_0 as a function of space, its topological properties are all derivable from such a physical realization of the field. Defining and constructing the properties constituting T is a formidable task but T is conceptually specific and unambiguous. By definition, two physical realizations of the same field, \mathbf{B}_1 and \mathbf{B}_2 , admissible in V under boundary condition (1) are topologically equivalent if one can be continuously deformed into the other, the same topological properties to be found in the two fields.

In a lecture, the goal of communication can be encumbered by the abstract and indirect T , a point that had arisen in our conversations. Continuous field deformation is the basic process whereas T is its derived property. Therefore, the Theorem should be stated directly in terms of the continuous deformations of the field without *a priori* involving T .

The statement should also avoid the assumption of \mathbf{B} being continuous in its force-free state described by equations (3) and (4). Therefore, we define equilibrium to be a state of energy extremum:

$$\delta\mathcal{E} \equiv \delta \int_V B^2 dV = 0 \quad (5)$$

for a specific continuous field \mathbf{B}_0 given in V subject to boundary condition (1). Thus, the Parker Magnetostatic Problem seeks a continuous deformation that takes the field in V from \mathbf{B}_0 to an equilibrium state defined by the extremum condition (5).

Induction equation (2) is a so-called Eulerian description of physical variables as functions of space-time, with no need at any time to know where a particular fluid particle is located. The fundamental property of this Eulerian description is better seen physically in the equivalent Lagrangian description stipulating that magnetic flux is conserved on every fluid surface identified and followed in its motion and deformation in time. The Lagrangian displacement of the fluid may be expressed as the continuous one-to-one map

$$\mathbf{x} = \mathbf{x}(\mathbf{x}_0), \quad (6)$$

giving the Cartesian location of each fluid particle initially located at \mathbf{x}_0 after a global, not necessarily infinitesimal, deformation of the fluid. Define the transformation matrix

$$\mathcal{T}_{ij} = \frac{\partial x_i}{\partial x_{0,j}}, \quad (7)$$

the determinant $|\mathcal{T}| \neq 0$ everywhere. From standard textbooks, an initial density $\rho_0(\mathbf{x}_0)$ of the fluid is transformed into $\rho(\mathbf{x})$ given by

$$\rho|\mathcal{T}| = \rho_0, \quad (8)$$

accounting for density compression during deformation. The precise distribution of ρ_0 has no physical significance for our discussion except for its use to describe the compressive effect of the deformation on the field that transforms as

$$\frac{B_i}{\rho} = \mathcal{T}_{ij} \left(\frac{B_{0,j}}{\rho_0} \right), \quad (9)$$

in terms of the Cartesian field components. Therefore, the Parker Magnetostatic Problem seeks a transformation $\mathcal{T}_{ij} : \mathbf{x} = \mathbf{x}(\mathbf{x}_0)$ that takes a prescribed $\mathbf{B}_0(\mathbf{x}_0)$ to $\mathbf{B}(\mathbf{x})$ under equation (9) to an equilibrium defined by extremum (5).

The Parker Magnetostatic Theorem can now be stated simply, that the equilibrium as an extremum (5) available to the arbitrarily prescribed $\mathbf{B}_0(\mathbf{x}_0)$ generally must embed TDs.

A note from the general theory of variational problems is in order. The existence of an extremum value of \mathcal{E} is separate from whether the exact extremum value can be realized in the space of *continuous* solenoidal vector-functions over which the magnetic energy \mathcal{E} is extremized; see the general discussion by Courant & Hilbert (1966, Vol. 1). In general, the extremum is not realized by any of the continuous fields admissible in V but may be approached to as close as desired in the function space. In the approach of the magnetic energy to an extremum, say, by constructing in an infinite sequence of extremizing fields, the fields may tend to a limit equilibrium field containing one or more TDs. The continuous deformation taking a field to that limit develops a singularity to create the TDs in the limit.

There is a continuum of $\mathbf{B}_0(\mathbf{x}_0)$ for which the extremum field is everywhere continuous and satisfies the force-free equations (3) and (4). For other prescriptions of $\mathbf{B}_0(\mathbf{x}_0)$, the extremum field is everywhere continuous, satisfying equations (3) and (4), except at inevitable TDs constructed as weak solutions in the following sense. Each TD is tangential with its external field and shaped such that magnetic pressure B^2 is continuous across it, represented by an additional discrete-current source in the form of a Dirac delta-function on the right side of force-free equation (3).

2. The Two-Plate Force-Free Fields. The 1972 Parker Theorem is based on examining the availability of continuous equilibrium states to the twisted fields in the infinitesimal, phase-space neighborhood of a uniform field, the fields anchored at a pair of parallel rigid plates. The treatment is limited to fields expressed as a small-parameter expansion about the uniform field accurate to all orders of the parameter, that suffices to make the fundamental point that, except for a physically unjustifiable restriction (z -invariance) on the twists one may endow on these neighborhood fields, equilibrium condition is incompatible with the field being spatially continuous. That is, most of the twisted fields in the neighborhood of the uniform field must achieve equilibrium with inevitable TDs.

In this context, the subset of *everywhere continuous* equilibria admissible in the two-plate problem are physically interesting as non-perturbational solutions to boundary value problems. Under astrophysical circumstances of a weak but nonzero resistivity, in the sense of a large magnetic Reynolds number, the TDs develop when and where the perfect-conductivity approximation applies but must dissipate as field gradients steepen unboundedly with the break down of the approximation. From this consideration we have two motivations for investigating the continuous force-free fields in the two-plate problem, one in recognizing these continuous states as end states following TD formation and dissipation. The other, as our study below shows, is that these continuous solutions give insight into inevitable TDs. The force-free states in the phase-space neighborhood of a current carrying field in the two-plate problem must be even richer in physical properties than those associated with the uniform field that led to the discovery of the Theorem.

Consider below an instructive constant- α force-free field anchored to the $z = \pm L$ plates, periodic in the (x, y) plane and invariant in z , that has a generalization to a non-uniform α field, both especially interesting in relation to the two-plate problem.

Taking α uniform in space, force-free equation (4) is satisfied. Represent the z -invariant field as:

$$\mathbf{B} = \frac{\partial A}{\partial y} \hat{\mathbf{x}} - \frac{\partial A}{\partial x} \hat{\mathbf{y}} + Q \hat{\mathbf{z}}, \quad (10)$$

$$\nabla \times \mathbf{B} = \frac{\partial Q}{\partial y} \hat{\mathbf{x}} - \frac{\partial Q}{\partial x} \hat{\mathbf{y}} - \nabla_{\perp}^2 A \hat{\mathbf{z}}, \quad (11)$$

and force-free equation (3) gives

$$Q(x, y) = \alpha A, \quad (12)$$

$$\nabla_{\perp}^2 A + \alpha^2 A = 0, \quad (13)$$

with the particular periodic solution

$$A = A_0 \sin k_1 x \sin k_2 y; \quad \alpha^2 = k_1^2 + k_2^2, \quad (14)$$

A_0 being an arbitrary constant amplitude and (k_1, k_2) the constant periodicity wave numbers.

Define $\alpha_0 = \sqrt{k_1^2 + k_2^2}$ and the solutions describes a pair of $\alpha = \pm \alpha_0$ fields with opposite magnetic twists:

$$\mathbf{B} = \frac{k_2}{\alpha_0} \sin k_1 x \cos k_2 y \hat{\mathbf{x}} - \frac{k_1}{\alpha_0} \cos k_1 x \sin k_2 y \hat{\mathbf{y}} + \frac{\alpha}{\alpha_0} \sin k_1 x \sin k_2 y \hat{\mathbf{z}}, \quad (15)$$

using the normalization $A_0 = \frac{1}{\alpha_0}$. B_x and B_y depend on α_0 but not on the sign of α , whereas B_z is independent of α_0 but of opposite signs for $\alpha = \pm \alpha_0$. By equation (3), the force-free current density flows parallel, or anti-parallel, to the field for $\alpha = \pm \alpha_0$, respectively.

We can construct three periodic force-free fields from the above simple solutions, each anchored at $z = \pm L$, the basic properties of which are shown in Figs. 1-3. Figs. 1 and 2 display two representative adjacent periodic-cells of the z -invariant fields for the cases of $\alpha = \pm\alpha_0$, respectively, each cell containing a flux rope winding from plate to plate. The z -axis points out of the figures. In Fig. 1, the $\alpha = \alpha_0$ field has $B_z > 0$ in the $0 < k_1x < \pi$ cell and $B_z < 0$ in the adjacent $\pi < k_1x < 2\pi$ cell. The directions of the transverse (x and y) field components along the boundary of each cell indicate both flux ropes are right-handedly twisted. In this case of $\alpha > 0$, J_z has the same algebraic sign as B_z in the two cells, J_z in each cell by Ampere's law consistent with the right-handed circulation of transverse field components. Fig. 2 similarly displays the $\alpha = -\alpha_0$ field, noting that both $\alpha = \pm\alpha_0$ fields have identical transverse field components, the corresponding cells of the two fields differing in the signs of B_z but having the same sign of J_z .

The $\alpha = \pm\alpha_0$ fields respectively satisfy the boundary conditions:

$$z = \pm L, \quad B_z = \pm \sin k_1x \sin k_2y, \quad (16)$$

for each periodic field the signs of B_z and J_z alternating from periodic cell to adjacent cell in both periodicity directions in the $x - y$ plane. The pattern of the signs of B_z of each field solution is not significant, because the pattern may be interpreted to have been imposed on the field by the *incidentally* prescribed boundary conditions. The cell-to-cell alternating sign of J_z is a general feature of periodic equilibria. Periodicity precludes any net current from plate to plate over length scales larger than the period, because a net current in the unbounded space $|z| < L$ has a logarithmically divergent magnetic energy.

Fig. 3 displays a periodic field composed of a $\alpha = \alpha_0$ right-handed flux rope (labeled Rope R) juxtaposed with a $\alpha = -\alpha_0$ left-handed flux rope (labeled Rople L) in adjacent periodic cells. This force-free field has a spatially varying α , jumping discontinuously in sign across each inter-cell boundary and satisfying the boundary conditions:

$$z = \pm L, \quad B_z = |\sin k_1x \sin k_2y|, \quad (17)$$

all flux ropes point in the same upward direction ($B_z > 0$ in each cell), entering the domain at $z = -L$ and leaving at $z = L$. In this field, J_z alternate in sign from cell to cell, with no net current from plate to plate on scales larger than the periodicity. A point about the discontinuity of α is noteworthy. Although the piecewise discontinuity of α across a cell boundary implies a discontinuity in the first derivatives of \mathbf{B} by force-free equation (3), it does not necessarily imply a discontinuity in \mathbf{B} as seen in the solution in Fig. 3.

Let us now return to the Parker two-plate problem. Starting with a uniform field $\mathbf{B}_1 = B_1\hat{\mathbf{z}}$ connecting $z = \pm L$, the rigidity of the boundaries is first relaxed to *continuously*

displace the foot-points (FTs) in some specific manner. The original Parker problem treated incompressible rotational FT displacements, so that the initial and displaced field satisfy the boundary conditions:

$$z = \pm L, \quad B_z = B_1 > 0, \quad (18)$$

to keep the problem simple. The FT displacements are continued smoothly into the fluid interior; the details of this continuation is not important. The global continuous deformation is described by transformation (9) for some \mathcal{T} that takes the field from $\mathbf{B}_1 = B_1 \hat{\mathbf{z}}$ in the interior $|z| < L$ to a deformed state denoted by $\mathbf{B}_0(\mathbf{x}_0)$. Now, restore the rigidity of the two plates and seek a second, further continuous transformation (9) that takes the field from $\mathbf{B}_0(\mathbf{x}_0)$ to $\mathbf{B}(\mathbf{x})$ at energy extremum (5) subject to boundary conditions (18).

For $\mathbf{B}_0(\mathbf{x}_0)$ differing infinitesimally from the uniform field $\mathbf{B}_1 = B_1 \hat{\mathbf{z}}$, described by a small parameter expansion, the Theorem shows that in the limit of $L \rightarrow \infty$, the force-free equations demand that the field must be invariant in z to all orders of the expansion, which is unjustifiably restrictive since the expansion permits arbitrary braiding of thin flux tubes carrying a irremovable z -dependence in $\mathbf{B}_0(\mathbf{x}_0)$. The Theorem then follows that the assumption of analyticity, the basis of the small parameter expansion, must fail if the field $\mathbf{B}_0(\mathbf{x}_0)$ is to be continuously deformed into an extremum (5) .

We may generalize the Parker problem to FT displacements to be compressible and rotational, which would change the boundary conditions on $\mathbf{B}_0(\mathbf{x}_0)$ to

$$z = \pm L, \quad B_z = F(x, y), \quad (19)$$

allowing for a variation of B_z on $z = \pm L$. By the conservation of magnetic flux across the boundaries $z = \pm L$, $F(x, y)$ must be positive definite. It follows that there is no way for a FT displacement to create boundary conditions (16) from boundary conditions (18). In contrast, the non-constant α field, Fig. 3, with no TD is accessible to a FT displacement from the uniform field, with B_z non-negative on $z = \pm L$ in boundary conditions (17).

The general property is worth a further discussion elsewhere that, a continuous FT displacement localized on $z = \pm L$ must generate equal and opposite currents flowing between the plates, so that the far field in $|z| < L$ sees no net current that would otherwise produce an azimuthal far field of a logarithmically divergent energy. Since the initial field is a, uni-directional, uniform field in the Parker two-plate problem, the force-free fields, with or without TDs, created in $|z| < L$ by FT displacements must be force-free with an α of opposite signs.

3. A Non-constant α Force-Free Field, better behaved. The piecewise-constant α field in Fig. 3 has an anomalous field structure that needs careful treatment as a product from FT displacement at $z = \pm L$. Along the inter-cell boundaries, $B_z = 0$, which means that these

boundaries are composed of closed field lines on the $x - y$ planes of constant z , not possible to produce by physically meaningful FT displacements from the uniform field. Moreover, in the neighborhoods the boundaries where B_z approaches zero, the field lines rise in height from $z = -L$ to circulate around the cell very many times before reaching $z = L$, the total number of circulations being unbounded as close to the boundary as a field line gets. We present a non-constant α generalization of this piecewise-constant α field that does not have this anomalous field topology.

Substitute equations (10) and (11) in the force-free equation

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = 0, \quad (20)$$

to obtain

$$\nabla_{\perp}^2 A \nabla A + Q \nabla Q + \frac{\partial(Q, A)}{\partial(x, y)} \hat{\mathbf{z}} = 0, \quad (21)$$

which implies

$$Q(x, y) = Q(A), \quad (22)$$

$$\alpha = \frac{dQ}{dA}, \quad (23)$$

$$\nabla_{\perp}^2 A + \frac{1}{2} \frac{d}{dA} Q^2(A) = 0. \quad (24)$$

Now set

$$Q^2 = Q_0^2 + \alpha_0^2 A^2, \quad (25)$$

$$\alpha = \frac{\alpha_0^2 A}{\sqrt{Q_0^2 + \alpha_0^2 A^2}}, \quad (26)$$

where Q_0 and α_0 are constants, from which the governing equation (24) takes the form

$$\nabla_{\perp}^2 A + \alpha_0^2 A = 0, \quad (27)$$

which brings us back to the linear PDE (13).

By the above analysis, we see that all three force-free fields in Figs. 1-3 may be generalized to fields of smoothly varying non-constant α (26), with no change in the transversed field components, but with $B_z = \pm \alpha_0 A$ replaced with $B_z = \pm \sqrt{Q_0^2 + \alpha_0^2 A^2}$ whereas the flux function A is governed by the same PDE (13). It is straightforward to see that the generalized solution $B_z = \sqrt{Q_0^2 + \alpha_0^2 A^2}$ meets the requirement of alternating signs in J_z from periodic cell to cell, and we now have the boundary conditions

$$z = \pm L, \quad B_z = \sqrt{Q_0^2 + \alpha_0^2 \sin^2 k_1 x \sin^2 k_2 y}, \quad (28)$$

the boundary flux positive definite and the force-free field without the infinite long field lines of the $Q_0 = 0$ special piecewise-constant force-free field.

The $Q_0 \neq 0$ force-free field is deformable from the uniform field $\mathbf{B}_1 = B_1 \hat{\mathbf{z}}$ without ending up with a TD in the force-free end state. What is more interesting is that further FT displacement imposed on the $Q_0 \neq 0$ force-free field with no respect for its very specialized form and breaking its z -invariance will produce a deformed field that can find equilibrium only by formation of TDs. At this point of the investigation, we have made available well behaved force-free fields which must transit into equilibrium states with TDs when suitably deformed with breaking of its intrinsic symmetry. There are more analytical solutions relating to the general constant- α fields treated in Conversation V and VII, but we have reached a good point of what would be pedagogically useful for introducing the Parker Theorem at the Crafoord Symposium.

I am ready to produce a draft PowerPoint viewgraph for Gene's consideration and approval.

Fig. 1

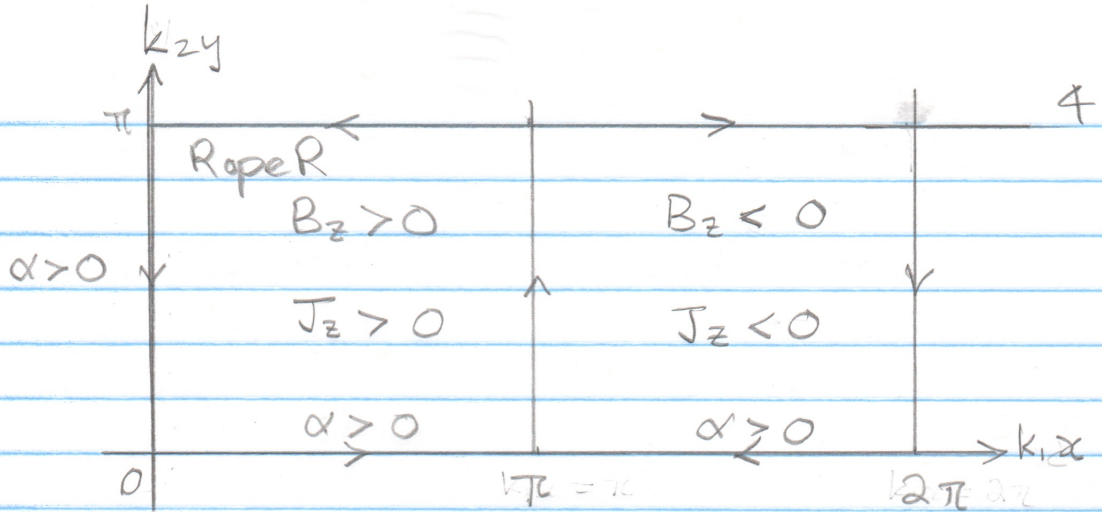


Fig. 2

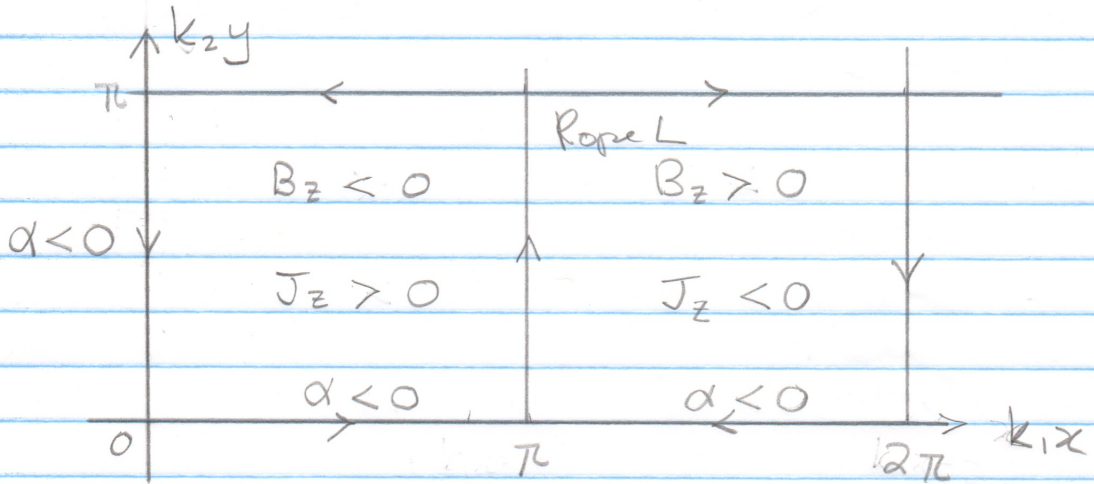
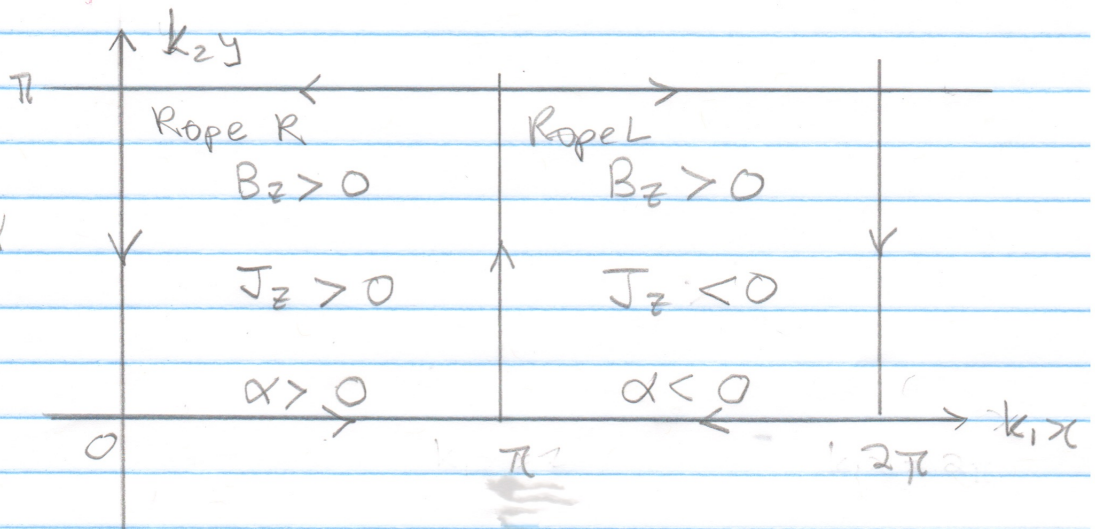


Fig. 3

Mixed α



Conversations with Eugene N. Parker¹

IX. Coupling the equations of ideal gas-dynamics to Maxwell's equations

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October 21, 2020

1. Coupling the equations of ideal gas-dynamics to Maxwell's equations

Consider the equations of ideal gas-dynamics

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p + \frac{1}{c} \mathbf{J} \times \mathbf{B} + \rho_e \mathbf{E}, \quad (1)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (2)$$

$$\frac{\partial}{\partial t} (p \rho^{-\gamma}) + \mathbf{v} \cdot \nabla (p \rho^{-\gamma}) = 0, \quad (3)$$

in the usual notations. The entropy function $S = p \rho^{-\lambda}$ is a Lagrangian constant on each infinitesimal fluid parcel. We have introduced the Lorentz and electrostatic forces, allowing for a charge density ρ_e , c being the speed of light. This fluid system is Newtonian invariant under Galilean transformation $(\mathbf{x}', t') = (\mathbf{x} - \mathbf{v}t, t)$, for any relative velocity \mathbf{v} between 2 moving frames of reference.

The Maxwell's equations are:

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \quad (4)$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}, \quad (5)$$

$$\nabla \cdot \mathbf{E} = 4\pi \rho_e, \quad (6)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (7)$$

invariant under Lorentz transformation,

$$\mathbf{x}'_{\parallel} = \gamma(\mathbf{x}_{\parallel} - \mathbf{v}t), \quad \mathbf{x}'_{\perp} = \mathbf{x}_{\perp}, \quad (8)$$

$$t' = \gamma \left(t - \frac{\mathbf{v} \cdot \mathbf{x}}{c^2} \right), \quad \gamma = \left(1 - \frac{v^2}{c^2} \right)^{-1/2}, \quad (9)$$

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where \mathbf{v} is the velocity of frame (\mathbf{x}', t') observed by frame (\mathbf{x}, t) , between which the electromagnetic fields transform as given by

$$\mathbf{E}' = \gamma \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right), \quad (10)$$

$$\mathbf{B}' = \gamma \left(\mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E} \right). \quad (11)$$

The two sets of equations are incompatible because of their distinct invariances, basic to our consideration here. The EM waves travel at the speed of light c , posing the question of which reference frame that speed is supposed to be observed. Special relativity resolves the question fundamentally for Maxwell's equations, by stating that c is a universal constant independent of frame. That is, Galilean invariance of physical laws is not fundamental but is an approximation in the Newtonian limit of $\frac{v^2}{c^2} \ll 1$. Thus, we have two regimes of physics. One is that of the Lorentz transformation for which Newtonian mechanics must and can be generalized to be compatible with Maxwell's equations, rigorously giving up Galilean invariance. The other is the Newtonian regime in which Maxwell's equations account for the Lorentz and electrostatic forces in momentum equation (1) in a manner consistent with Galilean invariance. A subtlety in derivation is to ensure that the speed of light c should not conceptually feature in the momentum equation. That is, the lowest order $\frac{v}{c} \ll 1$ expansion of Maxwell's equations must be examined carefully for their contributions to effects that are truly independent of c .

Under Galilean invariance, any two relatively moving frames see the same particle density since there is no Lorentz contraction in volume to speak off. A perfect conductor defined by zero \mathbf{E} in the rest frame of an electrically conducting fluid is meaningful in the $\frac{v^2}{c^2} \ll 1$ approximation that any electric field would have been shorted away at speeds taken to be infinite in this approximation. Equal and opposite electrical charges in the fluid produce a current density \mathbf{J} by the difference between their two streaming velocities. This difference in velocity is the same in any two Galilean frames, so both frames observe the same \mathbf{J} .

The differences in $(\mathbf{E}, \mathbf{B}, \mathbf{J}, \rho_e)$ of higher orders in $\frac{v}{c}$ must be neglected so that the governing fluid mechanics does not carry c explicitly, which would otherwise beg for the definition of c in terms of its observing frame of reference. We thus have a task of derivation, watching out, in particular, for higher-order effects that give rise to zero-order behaviors, independent of c , in Newtonian description.

We proceed with the Newtonian approximation of eqn (10) and (11)

$$\mathbf{E}' = \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} + \mathcal{O}\left(\frac{v^2}{c^2}\right), \quad (12)$$

$$\mathbf{B}' = \mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E} + \mathcal{O}\left(\frac{v^2}{c^2}\right). \quad (13)$$

Setting $\mathbf{E}' = 0$ for perfect conductivity yields

$$\mathbf{E} = -\frac{\mathbf{v}}{c} \times \mathbf{B}, \quad (14)$$

$$\mathbf{B}' = \mathbf{B}, \quad (15)$$

to first order in $\frac{v}{c}$. That is, the two frames observe the same magnetic field but a difference, of first order in $\frac{v}{c}$, in the electric field. We thus have the following consequences. Faraday's law takes the form of

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}), \quad (16)$$

the common factor of $\frac{1}{c}$ eliminated. We see that the electric-field, first order in $\frac{v}{c}$, gives rise to a zero-order property in the governing equation for \mathbf{B} . We regain Ampere's law,

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J}, \quad (17)$$

neglecting the displacement current of order $\frac{v^2}{c^2}$ in equation (5). We also see that ρ_e is of first order $\frac{v}{c}$, by which the electrostatic force in momentum equation (1) is of order $\frac{v^2}{c^2}$ and thus negligible. By Ampere's law, the Lorentz force in the momentum equation is expressed entirely in terms of the field \mathbf{B} , removing a factor $\frac{1}{c}$,

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p + \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B}, \quad (18)$$

which, together with equations (2), (3), (16) pose a complete hydromagnetic set of two vector equations and two scalar equations for $\mathbf{v}, \mathbf{B}, \rho, p$ as the primary quantities.

In this hydromagnetic description, all the other physical properties are secondary quantities derivable from the primary set, namely, the electric current \mathbf{J} , electric field \mathbf{E} , a higher-order effect that gives a zero-order effect on \mathbf{B} , and the temperature T by the ideal gas law,

$$p = 2 \frac{\rho}{m_p} k_B T, \quad (19)$$

in the illustrative case of fully-ionized hydrogen, m_p, k_B being the proton mass and Boltzmann constant. Eqns (14) and (6) suffice to justify neglecting the electrostatic force as negligible in the momentum equation as a higher-order effect.

In that connection, it should be noted that in Newtonian mechanics, density is independent of the reference frame. In relativistic mechanics, density and current density are not observed the same in different frames, complicated by Lorentz volume-contraction. Therefore, charge density ρ_e given by eqn (6) is not meaningful in the Newtonian sense, observed in the laboratory frame but not found in the rest frame of the fluid. Consistent with the requirement that all Newtonian effects have no conceptual role for the speed of light, we neglect

the electrostatic force in the momentum equation in the laboratory frame as a higher-order relativistic effect. To treat ρ_e self-consistently, other higher-order effects must be considered that takes the description into the relativistic regime. It seems that some plasma kinetic calculations combining Newtonian mechanics and the full Maxwell equations may be inherently inconsistent.

Conversations with Eugene N. Parker¹

X. Model of a heated, thermally-conducting, hydrostatic, optically-thin radiating atmosphere

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This model illustrates a general monotonic increase of temperature with height z in the upper region of a hydrostatically-stratified, plane parallel, thermally conducting, optically thin atmosphere, radiating by particle collisions and heated uniformly per unit mass by some source :

$$\frac{dp}{dz} + \rho g = 0, \quad (1)$$

$$-\frac{d}{dz} \left[\kappa(T) \frac{dT}{dz} \right] - \lambda \rho^2 T^\alpha + \gamma \rho = 0, \quad (2)$$

$$p = R_0 \rho T, \quad (3)$$

where p, ρ, T denote pressure, density and temperature with gas constant R_0 ; gravity is uniform with acceleration g ; thermal conduction is described by conductivity $\kappa(T)$; radiative loss with a constant amplitude λ and a constant power α dependence on temperature; heat addition per unit mass with a constant amplitude γ .

For convenience of derivation, we set $\kappa(T) = \frac{5}{2} \kappa_0 T^{5/2}$, κ_0 a constant, and $\alpha = 1$. Then, Eq (??) gives

$$\begin{aligned} \frac{d}{dz} \left[\kappa(T) \frac{dT}{dz} \right] &= \frac{dp}{dz} \frac{d}{dp} \left[\frac{5}{2} \kappa_0 T^{5/2} \frac{dp}{dz} \frac{dT}{dp} \right] \\ &= g^2 \rho \frac{d}{dp} \left[\frac{5}{2} \kappa_0 T^{5/2} \rho \frac{dT}{dp} \right]. \end{aligned} \quad (4)$$

By the elimination of a common factor ρ from Eq (??) and using ideal gas law (??), we get

$$\begin{aligned} \frac{g^2}{R_0} \frac{d}{dp} \left[\frac{5}{2} \kappa_0 T^{3/2} R_0 \rho T \frac{dT}{dp} \right] + \frac{\lambda}{R_0} R_0 \rho T - \gamma &= 0, \quad \text{ie,} \\ \frac{g^2}{R_0} \frac{d}{dp} \left(\kappa_0 p \frac{d}{dp} [T^{5/2}] \right) + \frac{\lambda}{R_0} p - \gamma &= 0, \end{aligned} \quad (5)$$

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which by a simple integration gives

$$\frac{g^2 \kappa_0}{R_0} p \frac{d}{dp} [T^{5/2}] + \frac{\lambda}{2R_0} p^2 - \gamma p + p_0 = 0, \quad (6)$$

p_0 an integration constant.

Rewriting Eq (??) as

$$\frac{g^2 \kappa_0}{R_0} \frac{d}{dp} [T^{5/2}] = -\frac{\lambda}{2R_0} p + \gamma - \frac{p_0}{p}, \quad (7)$$

a second integration gives

$$\frac{g^2 \kappa_0}{R_0} T^{5/2} = -\frac{\lambda}{4R_0} p^2 + \gamma p - p_0 \log \left(\frac{p}{p_1} \right), \quad (8)$$

where p_1 is an integration constant. The positive pressure $p(z)$ must decrease (ie, $\frac{dp}{dz} < 0$) z monotonically everywhere to support atmospheric weight, for example, decreasing to zero with $z \rightarrow \infty, p \rightarrow 0$ in an unbounded atmosphere ; ignoring, with no loss of generality, the also meaningful case of an atmospheric top located at a finite height $z = z_0 < \infty$. The temperature T being a positive function given by Eq (??) requires setting constants $p_0 > 0, p_1 > 0$ with $\frac{dT}{dp} < 0$ in the upper region $z \rightarrow \infty, p \rightarrow 0$; see Eq (??). That is, $\frac{dT}{dz} = \frac{dT}{dp} \frac{dp}{dz} > 0$ in the upper region where the temperature must monotonically increase with height.

Comments

1. The feature of temperature increasing with height is basic and general for a hydrostatic atmosphere. Stabilized with pressure and density decreasing with height, the lower atmosphere loses heat readily by radiation whereas overheating naturally occurs in the upper atmosphere, energy naturally balanced out by thermal conduction transporting the excess heat from the hot top to the cool bottom. The plane-parallel model treated is tightly gravitationally bounded, gravity being uniform in space. The temperature can never decrease with height above a critical height, as there is no way for the outward thermal flux to be removed.

2. The hydrostatic problem takes a more interesting form in the spherical space of a central star, for which gravity falls as r^{-2} in star-centered radial distance. In this case, we again consider an atmosphere extending to $r \rightarrow \infty$. Thermal energy can then be transported statically out to infinity, even without radiative cooling, by the r^{-2} spread of an expanding sphere. Force-balance has a critical issue, the further away from the star the weaker is gravity and thus the weaker is the pressure force required to support atmospheric weight.

The decrease of the pressure gradient $|\frac{dp}{dr}|$ with radial distance can readily be strong enough for the pressure to decrease asymptotically to a constant pressure (and a constant density) in a few stellar radii, the isothermal atmosphere being the canonical illustrative case. Provided this external infinite expense of uniform pressure and density can be justified, the conclusion is inescapable that hydrostatic equilibrium must break down somewhere in the near space of the star, taking the the far space of the star to be a vacuum. Thus, the high thermal conductivity of the solar corona is the hydrodynamic origin of the solar wind, whereas its high electrical conductivity accounts for the heating via the spontaneous formation and dissipation of current sheets.

3. The radiative loss $\lambda\rho^2T^\alpha$ with a constant α does capture real optically-thin radiative loss for a proper choice of the parameter α . We have chosen $\alpha = 1$ for the convenience of an analytically fully solvable problem. With $T(p)$ as an explicit solution given by Eq (??), ideal gas law Eq (??) defines $\rho(p)$ explicitly, from which $p(z)$ is given by force-balance Eq (??) a in simple quadrature.

4. In the plane-parallel hydrostatic model, how hot can the temperature be in the upper atmosphere for a given set of fluid properties, $\kappa_0, \alpha, \lambda, \gamma$? In the case of the corona, the high temperature gets to the level of $10^6 K$ at which the hydrogen gas is completely ionized, a distinct state from the lower atmosphere where neutral hydrogen as great radiators resides. This distinct state is drastic, the temperature transition layer from photospheric/chromospheric temperatures to the million degree temperature is only a few $10^2 km$.

Conversations with Eugene N. Parker¹

XI. The spherically-symmetric polytropic atmosphere: static equilibrium, steady stellar wind & time-dependent outflows

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A spherically-symmetric, polytropic static atmosphere in the unbounded domain $r > r_0$ has specific requirements for its gravitational confinement in vacuum, depending on the polytropic index γ . The analysis treats the property that the total mass of the static atmosphere has an upper limit if $\gamma < 4/3$. Any mass in excess of the limit must thus be shedded spontaneously and ejected out to infinity. Atmospheric heating naturally introduces mass into the atmosphere through its base. The upper atmosphere ejects its "excess" mass as the lower atmosphere heaves upward in near hydrostatic equilibrium, the global state naturally being that of a steady outflow.

In the fully ionized (mainly hydrogen) corona, the heating is magnetic in nature, maintaining its million-degree temperature at which thermal conduction is extremely efficient in delivering heat to the upper atmosphere. Moreover, the interstellar pressure is some 8 orders of magnitude smaller than typical coronal pressure. Thus the shedding of excess mass from the upper corona essentially is a free expansion into vacuum, the solar wind attaining supersonic speeds within 1-2 solar radii from the corona base, to blend way out in the tenuous interstellar medium via a terminal hydrodynamic shock beyond Pluto orbit.

The simple study presented here illustrates the mechanics of gravitational confinement of an atmosphere, leading to the mass limit theorem on the total atmosphere mass for the $\gamma < 4/3$ polytropes and a closed-form time-dependent solution for a $\gamma = 4/3$ polytrope whose internal energy density just suffices to sustain an expansion of an atmosphere, of a finite extent, all the way out to infinity.

1. The equilibrium atmosphere

We construct the solutions to the equilibrium equations:

$$\frac{dp}{dr} + \rho \frac{GM_0}{r^2} = 0, \quad (1)$$

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$$p = s_0 \rho^\gamma, \quad (2)$$

$$p = R_0 \rho T, \quad (3)$$

where p, ρ, T denote pressure, density and temperature with gas constant R_0 ; gravity owing to the central star of mass M_0 with Newton's constant G . Equation of state (2) directly relates p and ρ with constant polytropic index γ and entropy-like s_0 , as specifications of fluid properties. Eqns. (1) & (2) determine p and ρ as the primary quantities in terms of which the temperature T is a secondary quantity given by ideal gas law (3).

Eliminating p between Eqns. (1) & (2):

$$s_0(1+n) \frac{d}{dr} (\rho^{1/n}) = -\frac{GM_0}{r^2}, \quad (4)$$

where $\gamma = 1 + \frac{1}{n}$, we obtain the integral

$$\rho = \left(\frac{GM_0}{s_0 r_0 (1+n)} \right)^n \left(\frac{r_0}{r} + K_0 \right)^n, \quad (5)$$

where, with no loss of generality, the integration constant is written as $\frac{K_0}{r_0}$ in terms of the stellar radius r_0 . We then have

$$p = s_0 \left(\frac{GM_0}{s_0 r_0 (1+n)} \right)^{1+n} \left(\frac{r_0}{r} + K_0 \right)^{1+n}, \quad (6)$$

$$T = \frac{s_0}{R_0} \frac{GM_0}{s_0 r_0 (1+n)} \left(\frac{r_0}{r} + K_0 \right). \quad (7)$$

The constants r_0, s_0 and n are fixed properties of the system whereas the integration constant K_0 fixes the temperature $T = T_0$ at the atmospheric base:

$$\begin{aligned} T_0 &= \frac{s_0}{R_0} \frac{GM_0}{s_0 r_0 (1+n)} (1 + K_0) \\ &\equiv T_c (1 + K_0), \end{aligned} \quad (8)$$

defined relative to the critical temperature T_c . Eqns. (5), (6) & (7) show the following admissible ranges of K_0 . In the range $-1 < K_0 < 0$, the base temperature $0 < T_0 < T_c$ is sufficiently low for the atmospheric top to be located at finite radial distance $r_{top} = \frac{r_0}{|K_0|}$, where (p, ρ, T) vanish, abutting vacuum in $r > r_{top}$. In the special case $K_0 = 0$, the atmosphere extends to infinity with (p, ρ, T) monotonically decreasing to zero asymptotically, in particular, with $\rho \propto r^{-n}$. In the range $K_0 > 0$, the density, pressure & temperature all decrease with radial distance to asymptotic constant values beyond a few stellar radii.

The global radial extent of the atmosphere as a function of integration constant K_0 has a simple explanation. Pressure p must decrease with r to support the atmospheric weight. The polytropic equation of state and ideal gas law imply that ρ and T must also decrease with r . With less total mass to support at increasing r , it follows that $|\frac{dp}{dr}|$ must decrease with r . So steep is the decrease of pressure gradient with r in the case of the base temperature being below critical, $T_0 < T_c$, $-1 < K_0 < 0$, that the pressure and density hit zero at a finite atmospheric top $r_{top} = \frac{r_0}{|K_0|}$. At base temperature above critical, $T_0 > T_c$, $K_0 > 0$, the decrease of pressure gradient with r is gradual in the far reaches, where gravity is decreased in proportion to $1/r^2$, such that $p \rightarrow p_\infty = s_0 \left(\frac{GM_0 K_0}{s_0 r_0 (1+n)} \right)^{1+n} \neq 0$, $|\frac{dp}{dr}| \rightarrow 0$ asymptotically at $r \rightarrow \infty$.

2. The $K_0 < 0$ static solutions

The $K_0 < 0$ finite-extent atmospheres are the solutions to the theoretical problem of treating $r = r_0$ as a rigid wall and seeking the equilibrium state of a given mass M located in $r > r_0$. The mass achieves equilibrium by compressing against the wall under its own weight, the polytropic (equivalent to adiabatic) compression defining the temperature distribution as a function of r . For this problem, the mathematical tradition of boundary conditions for an ODE is conceptually awkward. It is meaningful to prescribe the total mass M so as to be specific about the physical system being considered. The total mass is free to be any value, M being an invariant of the fluid in $r > r_0$, whether in equilibrium or not. T_0 at $r = r_0$ is the base value of T under the specific condition of equilibrium, as determined by M . Thus, we should view the one free constant of the solution, the integration constant K_0 , to be parametrizing M :

$$\begin{aligned} M &= 4\pi \int_{r_0}^{r_{top} = \frac{r_0}{|K_0|}} \rho r^2 dr \\ &= 4\pi \left(\frac{GM_0}{s_0 r_0 (1+n)} \right)^n \int_{r_0}^{r_{top}} \left(\frac{r_0}{r} + K_0 \right)^n r^2 dr, \end{aligned} \quad (9)$$

with $K_0 < 0$. In other words, the given M determines K_0 as a root of integral (9) as a transcendental function of K_0 , with the root determining the equilibrium base temperature T_0 .

Taking $K_0 \rightarrow 0$, $r_{top} = \frac{r_0}{|K_0|} \rightarrow \infty$, and we have the limit:

$$M \rightarrow M_L = 4\pi \left(\frac{GM_0}{s_0(1+n)} \right)^n \lim_{r_{top} \rightarrow \infty} \int_{r_0}^{r_{top}} \frac{dr}{r^{n-2}} \quad (10)$$

$$= 4\pi \left(\frac{GM_0}{s_0(1+n)} \right)^n \frac{1}{(n-3)r_0^{n-3}}, \quad \text{if } n > 3. \quad (11)$$

The limit M_L is a finite number for the polytope if $n > 3$, unbounded if $n < 3$, and logarithmically divergent like $\log(r_{top})$ as $r_{top} \rightarrow \infty$ if $n = 3$.

We thus have the result on the polytropes with $n \leq 3$, in which case the integral (10) diverges, indicating that the total mass M may be prescribed as large as desired for which equilibrium exists with a finite radial extent, i.e., with $r_{top} < \infty$ and $K_0 \neq 0$.

The result on the polytropes with $n > 3$ is more interesting, in which case M_L is finite, implying that no equilibrium state is available for the total mass M if prescribed in excess of M_L . Conceptually, M is a freely prescribed entity. The conclusion is then unavoidable that enough mass must be spontaneously ejected out to infinity, leaving behind an equilibrium atmosphere containing the residual mass less than or equal to M_L . We will illustrate with an explicit time-dependent solution of such an expulsion in the last Section.

3. The physical meaning of the $K_0 > 0$ static solutions

The $K_0 > 0$ atmospheres extend to infinity where $(p, \rho, T) = (p_\infty, \rho_\infty, T_\infty)$ are asymptotically uniform. These solutions are physically distinct from the theoretical situation of chucking a (finite) total mass M into the otherwise vacuum exterior of the rigid wall $r = r_0$. The $K_0 > 0$ solutions solve a different theoretical problem, that of asking for the equilibrium state for an initially uniform, unbounded reservoir of fluid placed in $r > r_0$. Gravity would subsequently pull in a certain mass denoted by M_1 to press against to the rigid wall $r = r_0$. The initial uniform $(p, \rho, T) = (p_\infty, \rho_\infty, T_\infty)$ is preserved asymptotically outside of a few stellar radii. That is, the $K_0 > 0$ atmospheres are solutions to an accretion problem, for some mass M_1 in equilibrium around the rigid star contributed by the unbounded reservoir.

In this accretion problem, the meaning of integration constant $K_0 > 0$ is distinct from that of $-1 < K_0 < 0$. For a fixed s_0 as a property of the fluid in the reservoir, taking $r \rightarrow \infty$ in solution (5) gives

$$\rho_\infty = \left(\frac{GM_0}{s_0 r_0 (1+n)} \right)^n K_0^n, \quad (12)$$

showing that K_0 parametrizes the initially uniform density ρ_∞ of the reservoir, from which p_∞ and T_∞ are determined by Eqns. (2) and (3). It then follows that solution (5), (6) and (7) describes the global equilibrium of a total mass M_1

$$M_1 = 4\pi \int_{r_0}^{\infty} (\rho - \rho_\infty) r^2 dr$$

$$= 4\pi \left(\frac{GM_0}{s_0 r_0 (1+n)} \right)^n \int_{r_0}^{\infty} \left[\left(\frac{r_0}{r} + K_0 \right)^n - K_0^n \right] r^2 dr. \quad (13)$$

that has been ”pulled in” against the rigid wall $r = r_0$ as the excess of mass above the initially uniform reservoir. That is, integration constant K_0 parametrizes the uniform density of the initial reservoir that, of course, remains unchanged asymptotically at infinity and thus also fixes the additional mass M_1 pulled in by gravity from afar. The compression against the rigid wall $r = r_0$ also fixes the equilibrium base temperature T_0 .

Once the reservoir is prescribed with $(p, \rho, T) = (p_\infty, \rho_\infty, T_\infty)$ at infinity, fixed by the only free parameter K_0 in the system, the *equilibrium* accreted mass M_1 is uniquely determined. The atmosphere from $r = r_0$ to infinity is open with an unbounded mass. Mass can be moved to or from the far reaches of the reservoir to trap that unique mass M_1 in equilibrium.

4. The steady polytropic wind

Assuming a steady state and retaining the polytropic equation (2) and the ideal gas law (3), consider the hydrodynamic equations

$$\rho v \frac{dv}{dr} + \frac{dp}{dr} + \rho \frac{GM_0}{r^2} = 0, \quad (14)$$

$$r^2 \rho v = F_0, \quad (15)$$

describing a steady radial velocity with a constant mass flux F_0 , the domain taken formally to be $0 < r < \infty$ without bothering to connect the flow explicitly to the surface $r = r_0$. Substituting p for ρ by the polytropic equation, momentum equation (14) gives

$$\frac{d}{dr} \left(\frac{1}{2} v^2 + s_0 (1+n) \rho^{1/n} \right) = - \frac{GM_0}{r^2} \quad (16)$$

which integrates to give

$$\frac{1}{2} v^2 + s_0 (1+n) \rho^{1/n} = \frac{GM_0}{r_0} \left(\frac{r_0}{r} + K_0 \right), \quad (17)$$

again denoting the integration constant by K_0 . By mass conservation equation (15), we substitute v away to give an algebraic equation

$$\frac{F_0^2}{2r^4 \rho^2} + s_0 (1+n) \rho^{1/n} = \frac{GM_0}{r_0} \left(\frac{r_0}{r} + K_0 \right), \quad (18)$$

to solve for $\rho(r)$ in terms of which (p, T, v) can be expressed.

A summary of the polytropic stellar wind solutions will be inserted here at a later date, to be taken from EN Parker (1963), *Interplanetary Dynamical Processes*, Interscience.

The wind solutions must meet boundary conditions $r \rightarrow 0, v \rightarrow 0$ and $r \rightarrow \infty, v \rightarrow v_0$, the terminal speed v_0 carrying away the kinetic energy contributed from the fluid internal energy accelerating the flow from rest, less the work done to lift the weight of the mass in the wind.

The value of the polytropic index $\gamma = 1 + \frac{1}{n}$ is fundamental in this steady flow. The polytrope $n = 2, \gamma = 3/2$ is a critical value, corresponding to a uniform $v, \rho \propto r^{-2}, p \propto r^{-3}$, the weight of the fluid everywhere balanced exactly by the pressure force, each parcel of fluid coasting at the same constant velocity.

A polytropic wind accelerating from rest to a terminal speed exists only for $n \geq 2, \gamma \leq 3/2$, but not for $n < 2, \gamma > 3/2$ which includes the case of unheated adiabatic $\gamma = \gamma_0 = 5/3$ for an atmosphere composed of fully-ionized hydrogen such as the solar corona. This indicates that there would be no solar wind if the corona is an adiabatic gas. Heated to and maintained at its million-degree temperature, the corona is also an excellent thermal conductor. The corona thus behaves not as an adiabatic gas but more like a polytrope of $\gamma \approx 1$, with a thermal tendency for a uniform temperature distribution. Thus, the steady wind rather than a static equilibrium is the preferred state for the corona.

The criticality of $n = 2, \gamma = 3/2$, for whether the the polytropic atmosphere is static or steadily expanding as a wind, has a pretty explanation. With $n = 2, \gamma = 3/2$, the constant flow speed v is purely inertia because the weight of $\rho \propto r^{-2}$ in the the flow is *exactly* balanced by the polytropic pressure $p \propto r^{-3}$. Thus, the internal energy density p scales exactly with the gravitational binding-energy density $W \propto \rho \frac{Gm_0}{r} \propto r^{-3}$. If $n < 2, \gamma > 3/2$, a constant v flow with $\rho \propto r^{-2}$ has an internal energy $p \propto r^{-(1+\frac{1}{n})}$ declining with increasing r faster than the gravitational binding-energy density $W \propto \rho \frac{Gm_0}{r} \propto r^{-3}$, implying a decelerating rather than a steady flow. By a similar reasoning, if $n > 2, \gamma < 3/2$, the internal energy declines with increasing r less slowly than the gravitational binding-energy, ruling out a constant v for an accelerating v .

5. A time-dependent ejection from the $n = 3, \gamma = 4/3$ atmosphere

There is a scaling-law compatibility for time-dependent fluid expansion in a gravitational field analogous to the one governing the steady flow speed for the $n = 2, \gamma = 3/2$ polytrope.

Consider the scaling of pressure p and gravitational potential energy density $\frac{GM\rho}{r}$ in a

spherical ball of gas in homologous expansion centered at the source of gravity, in the case of the $n = 3, \gamma = 4/3$ polytrope. In orders of magnitude, the density scales with ball radius R as $\rho \propto R^{-3}$, the gravitational potential energy W scales as $W \propto \rho R^{-1} \propto R^{-4} \propto \rho^{-4/3}$ compatible with $p \propto \rho^{-4/3} \propto R^{-4}$. This compatibility points to the existence of self-similar solutions to the time dependent isentropic hydrodynamic equations:

$$\rho \left(\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} \right) + \frac{\partial p}{\partial r} + \rho \frac{GM_0}{r^2} = 0, \quad (19)$$

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \rho v) = 0, \quad (20)$$

$$\left(\frac{\partial}{\partial t} + v \frac{\partial}{\partial r} \right) (p \rho^{4/3}) = 0, \quad (21)$$

the temperature T defined by ideal gas law (3).

We derive from first principles the self-similar solutions found by Low (1982, *Self-similar magnetohydrodynamics. I. The $\gamma = 4/3$ polytrope and the coronal transient*. ApJ **254**, 796). Define the similarity variable

$$\zeta(r, t) = \frac{r}{\Phi(t)} \quad (22)$$

governing an expansion in the form of

$$\rho(r, t) = \frac{D(\zeta)}{\Phi^3(t)}, \quad (23)$$

$$p(r, t) = \frac{P(\zeta)}{\Phi^4(t)}. \quad (24)$$

Mass conservation equation (20) gives

$$-3\Phi^{-4} \frac{d\Phi}{dt} D(\zeta) + \Phi^{-3} \frac{dD}{d\zeta} \frac{\partial \zeta}{\partial t} + \Phi^{-4} \frac{1}{\zeta^2} \frac{\partial}{\partial \zeta} (\zeta^2 Dv) = 0. \quad (25)$$

Using $\frac{\partial \zeta}{\partial t} = -r\Phi^{-2} \frac{d\Phi}{dt} = -\zeta\Phi^{-1} \frac{d\Phi}{dt}$ to re-organize this equation, we have

$$\begin{aligned} -3\Phi^{-4} \frac{d\Phi}{dt} D(\zeta) - \Phi^{-4} \zeta \frac{dD}{d\zeta} \frac{d\Phi}{dt} + \Phi^{-4} \frac{1}{\zeta^2} \frac{\partial}{\partial \zeta} (\zeta^2 Dv) &= 0 \rightarrow \\ \frac{1}{\zeta^2} \frac{\partial}{\partial \zeta} \left(\zeta^2 D \left[v - \zeta \frac{d\Phi}{dt} \right] \right) &= 0, \end{aligned} \quad (26)$$

which permits setting the radial velocity to be

$$v = \zeta \frac{d\Phi}{dt} = \frac{r}{\Phi(t)} \frac{d\Phi}{dt}, \quad (27)$$

The polytropic entropy function

$$s(r, t) = p\rho^{-4/3} = P(\zeta)D^{-4/3}(\zeta), \quad (28)$$

being a strict function of ζ , entropy transport equation (21) is satisfied with no restriction on $P(\zeta)$ and $D(\zeta)$ by virtue of

$$\frac{\partial \zeta}{\partial t} + v \frac{\partial \zeta}{\partial r} \equiv 0, \quad (29)$$

with v given by (27). We return to the case of a uniform uniform polytropic entropy in the previous sections, setting

$$s = s_0, \quad (30)$$

a constant, with

$$\begin{aligned} p &= s_0 \rho^{-4/3} = s_0 \frac{D^{4/3}(\zeta)}{\Phi^4(t)}, \quad \text{or} \\ P(\zeta) &= s_0 D^{4/3}(\zeta). \end{aligned} \quad (31)$$

Momentum equation (19) then takes the form

$$D\zeta \frac{d^2 \Phi}{dt^2} + \left(\frac{dP}{d\zeta} + \frac{GM_0 D}{\zeta^2} \right) \Phi^{-2}(t) = 0, \quad (32)$$

which implies that

$$\frac{d^2 \Phi}{dt^2} = \frac{\alpha}{\Phi^2(t)}, \quad (33)$$

$$\frac{dP}{d\zeta} + \left(\frac{GM_0}{\zeta^2} + \alpha \zeta \right) D = 0, \quad (34)$$

where α is a free constant. The time-dependent flow may thus be viewed as a hydrostatic equilibrium in the non-inertial frame of reference (ζ, t) . The non-inertial transformation from physical space (r, t) to (ζ, t) -space gives rise to an body force proportional to ζ conceptually akin to the centrifugal force in the non-inertial rotational frame.

ODE (33) has the complete integral, with two integration constants t_0 and η ,

$$t + t_0 = \frac{1}{\eta} \sqrt{\Phi} \sqrt{\eta \Phi - 2\alpha} + I(\phi), \quad (35)$$

$$I = \frac{2\alpha}{V_0^2 \sqrt{V_0^2}} \log \left(\sqrt{V_0^2 \Phi} + \sqrt{V_0^2 \Phi - 2\alpha} \right), \quad \text{if } \eta = V_0^2 > 0, \quad (36)$$

$$I = \frac{2\alpha_0}{V_0^2 \sqrt{V_0^2}} \sin^{-1} \left(\sqrt{\frac{V_0^2 \Phi}{2\alpha_0}} \right), \quad \text{if } \eta = -V_0^2 < 0, \alpha_0 = -\alpha < 0, \quad (37)$$

where, with no loss of generality, we take $\alpha > 0$ in these formulae. A rich variety of temporal behaviors is generated by solution Φ .

It suffices for our purpose here to concentrate on the single case of $\alpha = 0$ with

$$\Phi = V_0(t + t_0), \quad (38)$$

V_0 being a free constant taken positive, thus removing the non-inertial force in equation (34) that reduces to

$$\frac{dP}{d\zeta} + \frac{GM_0}{\zeta^2} D = 0, \quad (39)$$

mathematically identical to the hydrostatic equation (1). The (Eulerian) velocity in this case is

$$v = \frac{r}{t + t_0}, \quad (40)$$

satisfying $\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} = 0$. Defining an initial state given at time $t = 0$, the solution describes an atmospheric expansion from this initial state with a spatially varying velocity, during which each parcel of fluid experiences no acceleration, since the net force is zero everywhere at all time. Most interesting is that this time-dependent flow transports both entropy and mass under gravity without upsetting the zero net force configuration, due fundamentally to the scaling compatibility between internal-energy density p and gravitational energy density $W \propto \rho \frac{GM}{r}$ in the case of the $\gamma = 4/3$ polytrope.

Equations (5), (6) & (7) give the $n = 3, \gamma = 4/3$ static equilibrium solution:

$$\rho = \left(\frac{4GM_0}{s_0 r_0} \right)^3 \left(\frac{r_0}{r} + K_0 \right)^3, \quad (41)$$

$$p = s_0 \left(\frac{4GM_0}{s_0 r_0} \right)^4 \left(\frac{r_0}{r} + K_0 \right)^4, \quad (42)$$

$$T = \frac{s_0}{R_0} \frac{4GM_0}{s_0 r_0} \left(\frac{r_0}{r} + K_0 \right), \quad (43)$$

that transforms into the self-similar time-dependent solution:

$$\rho = \left(\frac{4GM_0}{s_0 r_0} \right)^3 [V_0(t + t_0)]^{-3} \left(\frac{r_0}{\zeta} + K_0 \right)^3, \quad (44)$$

$$p = s_0 \left(\frac{4GM_0}{s_0 r_0} \right)^4 [V_0(t + t_0)]^{-4} \left(\frac{r_0}{\zeta} + K_0 \right)^4, \quad (45)$$

$$T = \frac{s_0}{R_0} \frac{4GM_0}{s_0 r_0} [V_0(t + t_0)]^{-1} \left(\frac{r_0}{\zeta} + K_0 \right), \quad (46)$$

that in physical spacetime reads

$$\rho = \left(\frac{4GM_0}{s_0 r_0} \right)^3 \left(\frac{r_0}{r} - \frac{K_1}{V_0(t+t_0)} \right)^3, \quad (47)$$

$$p = s_0 \left(\frac{4GM_0}{s_0 r_0} \right)^4 \left(\frac{r_0}{r} - \frac{K_1}{V_0(t+t_0)} \right)^4, \quad (48)$$

$$T = \frac{s_0}{R_0} \frac{4GM_0}{s_0 r_0} \left(\frac{r_0}{r} - \frac{K_1}{V_0(t+t_0)} \right), \quad (49)$$

taking $K_0 = -K_1 < 0$ for an expansive outflow in domain $0 < r < \infty$. This flow abuts vacuum at a moving boundary $r = r_{top}(t) = r_0 \frac{V_0(t+t_0)}{K_1}$ where (ρ, p, T) vanish, that travels at the constant speed $\frac{V_0}{K_1}$. The outflow at any fixed radial position slows down in time, as $t \rightarrow \infty$, to a static $\gamma = 4/3$ equilibrium atmosphere:

$$\rho = \left(\frac{4GM_0}{s_0 r_0} \right)^3 \left(\frac{r_0}{r} \right)^3, \quad (50)$$

$$p = s_0 \left(\frac{4GM_0}{s_0 r_0} \right)^4 \left(\frac{r_0}{r} \right)^4, \quad (51)$$

$$T = \frac{s_0}{R_0} \frac{4GM_0}{s_0 r_0} \frac{r_0}{r}, \quad (52)$$

In other words, the time-dependent mass flux across a chosen fixed boundary $r = r_0$, consistent with the global flow, slows down in time as $\frac{1}{t}$. Integrating over time, an unbounded amount of mass is spontaneously ejected out of the potential well of central, gravitating mass M_0 . The $\gamma = 4/3$ polytrope has just the right amount of thermal energy density p to not only sustain the flow against gravity but also to maintain zero net force throughout the flow. The more general case of the temporal function $\Phi(t)$ given by equations (35)-(37) includes accelerating and decelerating outflows for positive and negative constant α .

6. Coronal Mass Ejections & $\gamma = 4/3$ 3D hydromagnetic time-dependent flows

We return to the scaling compatibility between the $\gamma = 4/3$ internal energy p and gravitational energy $W = \frac{\rho GM}{R}$ in the sphere in homologous expansion. A magnetic field \mathbf{B} threading through the sphere, under the frozen-in condition of perfect electrical conductivity, must scale with the spherical expansion as $|\mathbf{B}| \propto R^{-2}$ to conserve magnetic flux. Magnetic pressure thus scales as $B^2 \propto R^{-4} \propto \rho^{4/3}$, compatible with the $\gamma = 4/3$ internal energy and gravitational energy. By this fundamental compatibility, the time-dependent, $\gamma = 4/3$ hydromagnetic equations

$$\rho \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B} - \nabla p - \rho \frac{GM}{r^2} \hat{\mathbf{r}}, \quad (53)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}), \quad (54)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (55)$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) (p \rho^{4/3}) = 0, \quad (56)$$

admit self-similar solutions of the form

$$\mathbf{v} = \frac{r}{\Phi^2(t)} \frac{d\Phi}{dt} \hat{\mathbf{r}}, \quad (57)$$

$$\rho = \frac{D(\zeta, \theta, \varphi)}{\Phi^3(t)}, \quad (58)$$

$$p = \frac{P(\zeta, \theta, \varphi)}{\Phi^4(t)}, \quad (59)$$

$$\mathbf{B} = \frac{\mathbf{H}(\zeta, \theta, \varphi)}{\Phi^2(t)}, \quad (60)$$

ζ being the similarity variable (22), $\Phi(t)$ describing the time-dependence given by equations (35)-(37). This class of solutions allows for fully 3D spatial variations expressed above in spherical coordinates (r, θ, φ) . The purely radial velocity (57) and the $\gamma = 4/3$ scaling laws render transport equations (54)-(56) satisfied with no restriction on (ρ, P, \mathbf{H}) as functions of (ζ, θ, φ) , whereas momentum equations (53) reduces to the magnetostatic equilibrium equation in similarity space:

$$\frac{1}{4\pi} (\nabla^* \times \mathbf{H}) \times \mathbf{H} - \nabla^* P - D \left(\frac{GM}{\zeta^2} + \alpha \zeta \right) \hat{\mathbf{r}} = 0, \quad (61)$$

again introducing the free constant α , the superscript * indicating gradients with respect to ζ . The existence of this class of 3D time-dependent hydromagnetic solutions reminds us that magnetic fields, especially if the field is in a great part not anchored to the base of the atmosphere, provide a source of energy quite compatible with both pressure and gravitational energy for a spontaneous, homologous expulsion of a 3D atmosphere out to infinity, a property exploited in the study of Coronal Mass Ejection (S. E. Gibson & B. C. Low 1998, *A time-dependent three-dimensional magnetohydrodynamic model of the Coronal Mass Ejection*, ApJ **493**, 460).

7. Concluding remarks

This Conversation XI lays out a background of 1D polytropic hydrodynamics to understand intuitively the nature of stellar atmospheres and their inevitable outward expansion.

The wind exists because the atmosphere is heated and thermally highly conducting, so that (i) excess mass is continually introduced into the atmosphere from below and (ii) significant thermal energy is conducted to the upper atmosphere, resulting in the excess mass being energized to a point of failing confinement in the radially declining gravity. Whereupon, the global wind-outflow takes over as the near-hydrostatic, lower atmosphere, by its heating, heaves outward to sustain the outflow. In the case of the Sun, the interstellar medium has a pressure some 8 orders of magnitude smaller than the pressure of the heated corona. Thus the outflow is a free expansion into essentially vacuum. The sound speed decreases along the outflow as the accelerated wind gathers speed, naturally becoming supersonic in a few stellar radii, which means that a terminal shock is to be expected in transition into the interstellar medium.

Conversations with Eugene N. Parker¹

XII. A formulation of the Parker Magnetostatic Theorem

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This is a study of the Parker Magnetostatic Theorem. The study is complete in its scope but there is much depth yet to fathom. The Parker two-plate magnetostatic problem is posed in terms of partial differential equations (PDEs) describing a force-free magnetic field expressed in Euler potentials with an explicit accounting for the field having been deformed continuously from a uniform field under the ideal induction equation. Our goal is to show that these PDEs are generally over-determining in the traditional undertaking of seeking *analytical* solutions to PDEs, a situation of more independent equations than the unknowns to be determined.

To break away from a common mindset, we start with the basic physical point that perfect electrical conductivity (zero electrical resistivity) conceptually admits magnetic tangential discontinuities (TDs or, equivalently, CSs). The magnetostatic PDEs restricted to analyticity are thus incomplete and must be extended to allow for the possible presence of TDs with Dirac- δ current densities. The TDs, in equilibrium, are mathematical singularities satisfying the governing PDEs in the integral sense. Equilibrium fields embedded with such singularities are examples of weak solutions in mathematical analysis.

Under astrophysical conditions of extremely high electrical conductivity, the chaotically-tangled 3D fields generally have no analytical equilibrium states available. The evolution of such a complex field thus readily and endlessly encounters TD formations and dissipation by the weak but nonzero electrical resistivity of a realistic embedding fluid. This basic theoretical conclusion was discovered by Parker (1972) as a Magnetostatic Theorem demonstrated with a small-parameter expansion rigorously taken to all orders. The novel derivation presented here is a general proof of the Parker Theorem by laying out the components of the Theorem to see directly that the equilibrium equations are analytically over determining.

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1. The force-free field in Euler-potential representation

The ordinary differential equations (ODEs)

$$\frac{dx}{B_x} = \frac{dy}{B_y} = \frac{dz}{B_z} \quad (1)$$

describe field lines in Cartesian space $(x, y, z) \equiv (x_1, x_2, x_3)$. Treating x and y as functions of z , the solution $x = f(z, U, V), y = g(z, U, V)$ of ODEs (1) describe each field line identified by fixed values of (U, V) as the two integration constants. Inverting the solution to express $U = U(x, y, z)$ and $V = V(x, y, z)$, the level surfaces of constant U and V are two independent families of foliated flux surfaces that mutually intersect along the field lines. The field lines are unique geometric objects in physical space, but their organization into two families of constant (U, V) flux surfaces is not unique. Just as any pair of independent functions $[F(U, V), G(U, V)]$ also serves just as well as a pair of integration constants of the ODEs, the level surfaces of constant F, G construct a family pair of flux surfaces on which all the field lines lie.

This neat representation of a field by Euler potentials (U, V) is always available in any localized region of space but, generally, not all such representations may be extended globally to an entire domain, requiring a construction of the mathematical connection between the Euler-potential representations in adjoining regions. It suits the purpose of our study to concentrate on a force-free field \mathbf{B} satisfying

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = 0, \quad (2)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (3)$$

the field represented by a pair of *global* Euler potentials:

$$\mathbf{B}(\mathbf{x}) = \nabla U \times \nabla V = \nabla \times (U \nabla V). \quad (4)$$

The gradients $(\nabla U, \nabla V)$ are respectively perpendicular to the flux surfaces of constant (U, V) , so their vector product is naturally parallel to \mathbf{B} as shown. Solenoidal condition (3) is trivially satisfied.

The Lorentz force is everywhere perpendicular to the field and force-free equation (2) gives

$$\begin{aligned} (\nabla \times \mathbf{B}) \times \mathbf{B} &= [\nabla \times (\nabla U \times \nabla V)] \times (\nabla U \times \nabla V) \\ &= \{[\nabla \times (\nabla U \times \nabla V)] \cdot \nabla U\} \nabla V \\ &\quad - \{[\nabla \times (\nabla U \times \nabla V)] \cdot \nabla V\} \nabla U \\ &= 0. \end{aligned} \quad (5)$$

The vectors ∇U and ∇V being independent, we set

$$[\nabla \times (\nabla U \times \nabla V)] \cdot \nabla U = 0, \quad (6)$$

$$[\nabla \times (\nabla U \times \nabla V)] \cdot \nabla V = 0. \quad (7)$$

By the vector identity

$$\nabla \cdot (\mathbf{P} \times \mathbf{Q}) = \mathbf{Q} \cdot \nabla \times \mathbf{P} - \mathbf{P} \cdot \nabla \times \mathbf{Q} \quad (8)$$

eqns. (6) and (7) can be rewritten as

$$\nabla \cdot [(\nabla U \times \nabla V) \times \nabla U] + (\nabla U \times \nabla V) \cdot \nabla \times (\nabla U) = 0, \quad (9)$$

$$\nabla \cdot [(\nabla U \times \nabla V) \times \nabla V] + (\nabla U \times \nabla V) \cdot \nabla \times (\nabla V) = 0. \quad (10)$$

The second terms on the left of these two equations vanish by vector identity, whereas the first terms can be further simplified to give

$$\nabla \cdot [(\nabla U \cdot \nabla V) \nabla U - |\nabla U|^2 \nabla V] = 0, \quad (11)$$

$$\nabla \cdot [(\nabla U \cdot \nabla V) \nabla V - |\nabla V|^2 \nabla U] = 0. \quad (12)$$

Eliminating ∇U and ∇V in turns, we obtain

$$\nabla \cdot [\{ |\nabla U|^2 |\nabla V|^2 - (\nabla U \cdot \nabla V)^2 \} \nabla U] = 0, \quad (13)$$

$$\nabla \cdot [\{ |\nabla U|^2 |\nabla V|^2 - (\nabla U \cdot \nabla V)^2 \} \nabla V] = 0, \quad (14)$$

as two coupled second-order PDEs for Euler potentials (U, V) subject to their boundary conditions.

This pair of PDEs for (U, V) can be recast to relate them directly to \mathbf{B} by first expressing equation (4) explicitly:

$$\mathbf{B}(\mathbf{x}) = \frac{\partial(U, V)}{\partial(y, z)} \hat{\mathbf{x}} + \frac{\partial(U, V)}{\partial(z, x)} \hat{\mathbf{y}} + \frac{\partial(U, V)}{\partial(x, y)} \hat{\mathbf{z}}. \quad (15)$$

Attaching subscripts to (U, V) to indicate partial differentiations with respect to (x, y, z) , we reduce the factor

$$\begin{aligned} & |\nabla U|^2 |\nabla V|^2 - (\nabla U \cdot \nabla V)^2 \\ &= \{U_x^2 + U_y^2 + U_z^2\} \{V_x^2 + V_y^2 + V_z^2\} - \{U_x V_x + U_y V_y + U_z V_z\}^2 \\ &= \{(U_x^2 V_x^2 + U_y^2 V_y^2 + U_z^2 V_z^2) + (U_x^2 V_y^2 + U_x^2 V_z^2 + U_y^2 V_z^2 + U_y^2 V_x^2 + U_z^2 V_x^2 + U_z^2 V_y^2)\} \\ &\quad - \{(U_x^2 V_x^2 + U_y^2 V_y^2 + U_z^2 V_z^2) + (2U_x V_x U_y V_y + 2U_y V_y U_z V_z + 2U_z V_z U_x V_x)\} \end{aligned}$$

$$\begin{aligned}
&= U_x V_y (U_x V_y - U_y V_x) - U_x V_z (U_z V_x - U_x V_z) + U_y V_z (U_y V_z - U_z V_y) \\
&\quad - U_y V_x (U_x V_y - U_y V_x) + U_z V_x (U_z V_x - U_x V_z) - U_z V_y (U_y V_z - U_z V_y) \\
&= U_x V_y \frac{\partial(U, V)}{\partial(x, y)} - U_x V_z \frac{\partial(U, V)}{\partial(z, x)} + U_y V_z \frac{\partial(U, V)}{\partial(y, z)} \\
&\quad - U_y V_x \frac{\partial(U, V)}{\partial(x, y)} + U_z V_x \frac{\partial(U, V)}{\partial(z, x)} - U_z V_y \frac{\partial(U, V)}{\partial(y, z)} \\
&= \frac{\partial(U, V)}{\partial(x, y)} (U_x V_y - U_y V_x) + \frac{\partial(U, V)}{\partial(y, z)} (U_y V_z - U_z V_y) - \frac{\partial(U, V)}{\partial(z, x)} (U_z V_x - U_x V_z) \\
&= \left[\frac{\partial(U, V)}{\partial(x, y)} \right]^2 + \left[\frac{\partial(U, V)}{\partial(y, z)} \right]^2 + \left[\frac{\partial(U, V)}{\partial(z, x)} \right]^2 \\
&= B_x^2 + B_y^2 + B_z^2 = B^2, \tag{16}
\end{aligned}$$

the last step by equation (15). Therefore, PDEs (13) and (14) may be expressed tidily as

$$\nabla \cdot (B^2 \nabla U) = 0, \tag{17}$$

$$\nabla \cdot (B^2 \nabla V) = 0, \tag{18}$$

subject to boundary conditions on (U, V) .

To summarize, it takes 2 scalar functions to define the three components of a magnetic field, noting that the third field component is fixed in terms of the other two under the solenoidal condition. Provided the Euler potentials (U, V) are globally defined, they are a natural choice of a pair of scalar functions to define the force-free field, governed by the two nonlinear PDEs (13) and (14) that is reducible to the equivalent pair (17) and (18).

2. The nature of the UV static boundary-value problem

2.1. General consideration

To discuss in general terms, consider a perfectly-conducting fluid embedding a magnetic field in a simply-connected domain D with its non-vanishing, normal component of the field

$$B_n(\partial D) = \mathbf{B} \cdot \hat{\mathbf{n}} \neq 0 \tag{19}$$

prescribed on its rigid, perfectly conducting boundary ∂D . When not in equilibrium, the field \mathbf{B} and velocity \mathbf{v} evolve by the induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}). \tag{20}$$

Rigidity of ∂D implies $v_n = \mathbf{v} \cdot \hat{\mathbf{n}} = 0$ on ∂D . Continuity of the tangential electric field at ∂D , a requirement of Maxwell's equations, demands that the boundary tangential velocity must also vanish by virtue of $B_n(\partial D) \neq 0$, leading to the non-slip boundary condition on the velocity

$$\mathbf{v}(\partial D) = 0, \quad (21)$$

which then implies that magnetic boundary condition (19) is fixed for all time.

Consider a non-equilibrium field $\mathbf{B}_{neq}(\mathbf{x})$ that is *spatially continuous* everywhere in D , which defines

$$B_n(\partial D) = B_{neq,n}(\partial D). \quad (22)$$

Quite separate from the specific momentum equation generating the velocity, a field evolving from $\mathbf{B}_{neq}(\mathbf{x})$ given at time $t = t_0$ is governed by the induction equation with a velocity that satisfies the non-slip boundary condition (21) and preserves the boundary distribution $B_n(\partial D)$. Denote by $\mathcal{S}[B_n(\partial D), \tau]$ the space of *continuous*, solenoidal vector distributions all sharing the same $B_n(\partial D)$. By definition the field $\mathbf{B}_{neq}(\mathbf{x})$ belongs to $\mathcal{S}[B_n(\partial D), \tau]$ and it evolves in time with velocity \mathbf{v} under the induction equation as a continuous sequence of fields in $\mathcal{S}[B_n(\partial D), \tau]$.

The basic property of the induction equation is the conservation of magnetic flux across every fluid surface σ deforming in the flow,

$$\frac{d}{dt} \mathcal{F} \equiv \frac{d}{dt} \int_{\sigma} \mathbf{B} \cdot \hat{\mathbf{n}} d\sigma = 0, \quad (23)$$

from which follows that the fluid is permanently partitioned by \mathbf{B} into a continuum of magnetic flux-tubes of fluid and layers of fluids between pairs of magnetic flux surfaces. The topology of a given set of geometric objects by textbook definition comprises all those properties that are invariant under all continuous deformations of these objects, e.g., the entanglement between two flux tubes each anchored rigidly at its two ends on ∂D . Any two fields in $\mathcal{S}[B_n(\partial D), \tau]$ are deformable continuously one into the other under the induction equation. Thus, the notation τ indicates that $\mathcal{S}[B_n(\partial D), \tau]$ is the function space of all topologically equivalent fields.

Again noting that our consideration is separate from the momentum equation generating \mathbf{v} , we have the interesting kinematic problem of determining a field in $\mathcal{S}[B_n(\partial D), \tau]$ that has an extremum in total magnetic energy,

$$\delta \mathcal{E} \equiv \delta \int_D B^2 dV = 0. \quad (24)$$

The force-free equation (2) is in context here the Euler-Lagrangian equations of this variational problem.

As a note of interest, define $\mathcal{S}_{all}[B_n(\partial D)] = \bigcup_{\tau} \mathcal{S}[B_n(\partial D), \tau]$, the union of all spaces of different topology τ . The extremum (24) taken over $\mathcal{S}_{all}[B_n(\partial D)]$ identifies the unique potential field \mathbf{B}_{pot} defined by $B_n(\partial D)$ with energy \mathcal{E}_{pot} which is readily shown to be the field admissible in D that has the absolute minimum total energy.

We shall see that constraining the extremum to be taken over a subspace $\mathcal{S}[B_n(\partial D), \tau] \subset \mathcal{S}_{all}[B_n(\partial D)]$ of a chosen topology τ presents the following complication common to variational problems (Courant & Hilbert 1963). The existence of a minimum of \mathcal{E} over any vector space \mathcal{S} must be distinguished from the possible actual realization of that minimum value by a vector in \mathcal{S} . Variational methodology is based on the assumption of field continuity to employ differential calculus whereas the physics of the problem may not be compatible with this mathematical convenience. The existence of \mathcal{E}_{pot} as the absolute minimum implies that a minimum \mathcal{E}_{min} must exist in each vector space $\mathcal{S}[B_n(\partial D), \tau]$, without necessarily implying that \mathcal{E}_{min} is realized in an actual field in $\mathcal{S}[B_n(\partial D), \tau]$. Quite generally, an infinite sequence of \mathbf{B} in $\mathcal{S}[B_n(\partial D), \tau]$ may be constructed whose respective total energies decrease and converge to \mathcal{E}_{min} without realizing this value within the space. Mathematical analysis calls function spaces Cauchy complete if all infinite sequences in them do converge to a limit within the given function space. The mathematical fact is that function spaces are generally Cauchy incomplete.

We end this subsection with a simple geometric example illustrating Cauchy incompleteness. Consider the space of all regular polygons of integer N equal sides with corners all lying on a unit circle. The lengths of the polygonal perimeters obviously have a maximum given by the length of the circumference of the unit circle. This maximum cannot be attained but can be approached as close as desired by any sequence of polygons of increasing $N \rightarrow \infty$.

The space of the polygons is analogous to an open unit interval $0 < x < 1$ on the line of real number x . The example seems to be just a matter of using the right language, but in the more complex situation of the force-free field as a problem in extremum variation calculus, whether or not a function space is Cauchy complete has fundamental physical consequences. That $\mathcal{S}[B_n(\partial D), \tau]$ is generally not Cauchy complete means that its minimum energy state is realized in a *discontinuous* field not in $\mathcal{S}[B_n(\partial D), \tau]$, not entirely unexpected because magnetic tangential discontinuities (TDs) with discrete current sheets (CSs) are conceptually a basic property of the perfect conductor with zero electrical resistivity. That TDs may arise is not new under some well known physical circumstances, like the TD forming at a 2D X-type magnetic neutral point in classical journal literature. The general inevitability of magnetic tangential discontinuities (TDs) in 3D fields, for *most* topology τ is a far reaching fundamental property first discovered as a mathematical theorem by Parker (1972).

2.2. Fields with global (U, V) Euler potentials

In terms the global Euler potentials (4), induction equation (20) reduces to

$$\frac{dU}{dt} \equiv \left[\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right] U = 0, \quad (25)$$

$$\frac{dV}{dt} \equiv \left[\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right] V = 0, \quad (26)$$

showing that the moving fluid continuously redistributes (U, V) in the interior of D with the values of (U, V) permanently fixed on each fluid element, whereas the boundary values of (U, V) are stationary under the no-slip boundary condition (21). Given a non-equilibrium field $\mathbf{B}_{neq}(x, y, z)$ at time $t = t_0$, its pair of Euler potentials $[U_{neq}(x, y, z), V_{neq}(x, y, z)]$ are the solutions to field line ODEs (1). The evolution of this field by any velocity \mathbf{v} under the induction equation may then be described by the solutions (U, V) to advection equations (25) and (26) subject to the initial-boundary conditions:

$$[U, V] = [U_{neq}(x, y, z), V_{neq}(x, y, z)] \quad \text{in } D \text{ at } t = t_0, \quad (27)$$

$$[U, V] = [U_{neq}(x, y, z), V_{neq}(x, y, z)] \quad \text{at } \partial D \text{ for } t > t_0. \quad (28)$$

Boundary condition (28) implies that the boundary normal field-component is stationary on ∂D ,

$$\begin{aligned} B_n(\partial D) &= [(\nabla U \times \nabla V) \cdot \hat{\mathbf{n}}]_{\partial D} \\ &= [(\nabla U_{neq} \times \nabla V_{neq}) \cdot \hat{\mathbf{n}}]_{\partial D} \\ &= [\nabla U_{neq} \cdot (\nabla V_{neq} \times \hat{\mathbf{n}})]_{\partial D}, \end{aligned} \quad (29)$$

which is independent of time because B_n is defined entirely in terms of the stationary Euler potentials and their tangential derivatives on ∂D .

Of interest to the 1972 Parker Theorem are the evolutions of (U, V) driven along paths of monotonically decreasing total energy \mathcal{E} in vector space $\mathcal{S} [B_n(\partial D), \tau]$ to arrive at a minimum \mathcal{E} equilibrium state for which the vanishing of the first order variation (24) is a necessary condition. Physically, the velocity \mathbf{v} is determined by the coupling of the induction equation to a specific momentum equation. To keep our discussion strictly kinematical and simple, it suffices to point out an example of a path of monotonically decreasing \mathcal{E} . Consider a viscous fluid so tenuous that there are only the Lorentz and viscous forces. The Lorentz force resisted by friction irreversibly convert magnetic energy into kinetic energy for viscous dissipation to arrive at a static, $\delta\mathcal{E} = 0$ force-free state. In terms of the Euler potentials

(15), variational equation (24) takes the form

$$\begin{aligned} \delta\mathcal{E} &= \delta \int_D \left\{ \left[\frac{\partial(U, V)}{\partial(x, y)} \right]^2 + \left[\frac{\partial(U, V)}{\partial(y, z)} \right]^2 + \left[\frac{\partial(U, V)}{\partial(z, x)} \right]^2 \right\} dV \\ &= \delta \int_D \{ |\nabla U|^2 |\nabla V|^2 - (\nabla U \cdot \nabla V)^2 \} dV = 0, \end{aligned} \quad (30)$$

making use of equation(16). PDEs (9) and (10) are the Euler-Lagrangian equations determining the (U, V) force-free field subject to Dirichlet boundary conditions (28).

This (U, V) Dirichlet boundary value problem is equivalent to the direct boundary value problem in terms of \mathbf{B} . The relationship between the two is subtle but essential to understanding the Parker Theorem.

Staying within the generality of discussion in this section, the direct problem demands for a force-free field in D subject to a prescribed $B_n(\partial D)$, not any force-free field but one that is deformable to a reference non-equilibrium field \mathbf{B}_{neq} . Note that there is a continuum of non-equilibrium fields in D topologically equivalent to a given \mathbf{B}_{neq} , so the reference field is not a unique field but can be any one of these equivalent fields. Rewrite force-free equations as

$$\nabla \times \mathbf{B} = \alpha \mathbf{B}, \quad (31)$$

$$\mathbf{B} \cdot \nabla \alpha = 0, \quad (32)$$

introducing the usual proportionality factor α . There is a continuum infinity of force-free fields sharing the fixed $B_n(\partial D)$, generated by α as a free function of space constrained by equation (32) to be a constant along each field line. We do not have a traditional boundary problem in PDEs. Instead, we have a problem of identifying a reference field \mathbf{B}_{neq} with one or more solutions among the many solutions of PDEs (31) and (32) subject only to boundary condition (19). The challenge of our problem is in the explicit implementation of \mathbf{B}_{neq} in the construction of a solution.

We may restrict the force-free fields to be analytic, populating a space of continuous equilibrium fields we denote by $\mathcal{B}_{\text{ff}}[B_n(\partial D)]$. The Parker Theorem then poses the problem of whether for every reference field \mathbf{B}_{neq} of a certain topology τ , there is a continuous force-free field $\mathbf{B}_{\text{ff}} \in \mathcal{B}_{\text{ff}}[B_n(\partial D)]$ attainable by a continuous deformation of \mathbf{B}_{neq} . Since perfect electric conductivity admits equilibrium-TDs, the vector-function space $\mathcal{B}_{\text{ff}}[B_n(\partial D)]$ is a sub-space of a larger vector-function space $\mathcal{B}_{\text{ff}}^*[B_n(\partial D)]$ containing continuous force-free fields with or without embedded equilibrium-TDs. The Parker Magnetostatic Problem can be posed more precisely, whether there are field topologies τ for which a given reference field \mathbf{B}_{neq} has no

topological equivalent $\mathbf{B}_{\text{ff}} \in \mathcal{B}_{\text{ff}}[B_n(\partial D)]$, implying that \mathbf{B}_{neq} must then attain a force-free state embedded with equilibrium-TDs.

The Parker Problem in terms of (U, V) is a traditional Dirichlet boundary-value problem that implements an explicit but incomplete topological role for the reference field \mathbf{B}_{neq} . The given \mathbf{B}_{neq} subject to a fixed $B_n(\partial D)$ has a pair of Euler potentials $(U_{\text{neq}}, V_{\text{neq}})$, as solutions to the field-line ODEs (1), that naturally are consistent with $B_n(\partial D)$ given by equation (29). Dirichlet boundary conditions (28) for (U, V) on ∂D then go on to determine a (U, V) force-free field as a solution of the Euler-Lagrangian PDEs (9) and (10). Let us restrict our interest to those fields with field lines of constant (U, V) all anchored at both ends at ∂D . Then, boundary conditions (28) is a one-to-one map of ∂D onto itself giving the connectivity of each field line by its pair of foot-points.

To proceed, a clarification is essential. If we chose to formulate the above Dirichlet boundary value problem using a different, topologically equivalent reference field \mathbf{B}_{neq} matching the fixed $B_n(\partial D)$, its different pair of Euler potentials $(U_{\text{neq}}, V_{\text{neq}})$ would still consistently give the fixed $B_n(\partial D)$ under equation (29) and give the same foot-point map from ∂D onto itself by the simple reason that all references fields are deformable, by definition, one into another.

We are dealing with a Dirichlet boundary-value problem governed by nonlinear elliptic PDEs (9) and (10), the solutions of which may not exist and, if they do exist, may not be unique. The magnetic connectivity of ∂D onto itself, implied by the boundary (U, V) distributions on ∂D , in fully 3D fields, has been built into the force-free field to be determined inside the domain D . Therefore if a solution exists, it will have the the built-in connectivity, but the actual topology of the field denoted by τ , which is more than just foot-point connectivity, has no role in the interior force-free, point-wise relationship between (U, V) . The possibility is thus presented that the interior field as a solution to the (U, V) Dirichlet boundary-value problem may not exhibit the topology τ , which is to say, the force-free field as a solution satisfying boundary conditions (28) is not deformable to the reference field \mathbf{B}_{neq} . The Parker Theorem presents and demonstrates a more interesting reality of this problem, that for most topologies τ , discontinuous solutions, the weak solutions with equilibrium-TDs, are the proper force-free states available to an arbitrarily prescribed \mathbf{B}_{neq} .

We are thus led to a need to implement the full topology τ as an explicit condition to be coupled to the (U, V) Dirichlet boundary-value problem. The (U, V) Dirichlet boundary-value problem is mathematically adequately posed under the assumption of analyticity. Either a an analytical solution exists or it does not, the problem is adequately posed in the sense that the questions of existence and uniqueness of solutions are meaningful and can be addressed unambiguously. To built the mathematically extraneous condition of ensuring the

right topology is found in the force-free field determined by the Dirichlet boundary-value problem, we end up with an over determining set of PDEs, more equations than there are the unknowns, which is not forbidden but compatibility among the governing PDEs is an essential issues. We then expect the Parker Theorem to present the possibility that for certain topologies τ , the additional conditions to ensure τ can be met with the continuous fields of the sub-space $\mathbf{B}_{\text{ff}} \in \mathcal{B}_{\text{ff}}[B_n(\partial D)]$.

Now we are ready to treat the Parker two-plate magnetostatic problem where our task is to build the aforementioned over-determined system of PDEs; yet to be finished. The preliminary result has discovered a generalized *z-invariance* requirement discovered in the Parker 1972 study based on an ϵ -power series.

3. The Parker two-plate problem

3.1. Original statement of problem, 1972

Consider a uniform field $\mathbf{B} = B_0\hat{\mathbf{z}}$ in domain $0 < z < L$ containing a perfectly conducting fluid, with rigid, perfectly conducting boundaries $z = 0, L$. Prepare a non-equilibrium field \mathbf{B}_{neq} by deforming the uniform field under the induction equation. First relax wall rigidity at $z = 0$ to displace the magnetic foot-points continuously and tangentially along $z = 0$. Let the foot-point displacement be incompressible, so that B_z remains uniform and unchanged on $z = 0$, as the prescribed boundary displacement continues into the fluid interior to deform the uniform field into \mathbf{B}_{neq} . Now restore wall rigidity at $z = 0$. The further deformation of the field from the non-equilibrium state \mathbf{B}_{neq} under the induction equation is then subject to boundary conditions

$$B_z = B_0 \quad \text{at } z = 0, L, \tag{33}$$

$$\mathbf{v} = 0 \quad \text{at } z = 0, L. \tag{34}$$

These boundary conditions permanently lock the three-dimensional twists, braidings and entanglements created in the field \mathbf{B}_{neq} by the imposed foot-point displacement. Once locked, the topological measures of these properties are invariant under all further continuous deformation.

These invariant measures are conceptually specific by the textbook definition of topology although identifying and explicitly expressing the measures is a technical problem challenging in its own right. Two fields in $0 < z < L$ are topologically equivalent if one can be deformed into the other continuously under boundary conditions (33) and (34), an operational test of equivalence that conveniently avoids dealing with the topological measures directly.

The history of the continuous, foot-point displacement alone creates the topology locked into \mathbf{B}_{neq} . The specific manner in which a given boundary displacement is continued into the fluid domain is not topologically constraining. Different continuations into the fluid domain from the same boundary displacement give different non-equilibrium \mathbf{B}_{neq} that are deformable one into another under boundary conditions (33) and (34). The continuum of topologically equivalent fields populate a space $\mathcal{S}[\tau]$ of continuous fields, denoting the common topology by τ . In general, this space does not include the uniform field, meaning that a field $\mathbf{B}_{neq} \in \mathcal{S}[\tau]$ generally cannot be deformed into the uniform field. The more interesting question posed by the Parker magnetostatic problem is whether this space contains a continuous force-free field \mathbf{B}_{ff} into which any $\mathbf{B}_{neq} \in \mathcal{S}[\tau]$ may be continuously deformed.

We proceed to a proof of the Theorem by that for most topology τ , the space $\mathcal{S}[\tau]$ is not Cauchy-complete wherein the extremum in the total magnetic energy can be approached only in the limit of a force-free field containing equilibrium TDs.

Note that the velocity in induction equation (20) has a free component everywhere parallel to the field that has no influence on the evolving field. For the field anchored to boundaries $z = 0, L$, entirely as a kinematic consideration, any velocity driving the field evolution can satisfy the rigid-wall boundary condition of $\mathbf{v} \cdot \hat{\mathbf{n}} = 0$ via a suitable prescription of its free field-aligned component.

3.2. Lagrangian field transform, Euler potentials, and Parker optical analogy

The end result of any continuous deformation of the fluid is a one-to-one Lagrangian transformation

$$\mathbf{x}' = \mathbf{x}'(\mathbf{x}), \quad (35)$$

$$T_{ij} = \frac{\partial x'_i}{\partial x_j}, \quad (36)$$

$$|T| = \epsilon_{ijk} \frac{\partial x'_1}{\partial x_i} \frac{\partial x'_2}{\partial x_j} \frac{\partial x'_3}{\partial x_k} \neq 0, \quad (37)$$

describing the net displacement of each fluid element from its Cartesian location $\mathbf{x} = (x_1, x_2, x_3)$ to Cartesian location $\mathbf{x}' = (x'_1, x'_2, x'_3)$. Continuity of deformation implies that the infinitesimal neighborhood of every fluid element in \mathbf{x} -space is transformed onto an infinitesimal neighborhood of the fluid element in \mathbf{x}' -space. Thus, different histories of continuous deformations are topologically equivalent if they produce at the end the same Lagrangian transformation of the fluid elements.

By the mass-conservation equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (38)$$

the density distribution transforms as

$$\rho'(\mathbf{x}') = \frac{\rho(\mathbf{x})}{|T|}. \quad (39)$$

Combining the induction and mass-conservation equations, we have the field transform

$$\frac{B'_i(\mathbf{x}')}{\rho'(\mathbf{x}')} = T_{ij} \frac{B_j(\mathbf{x})}{\rho(\mathbf{x})}, \quad (40)$$

which gives

$$B'_i(\mathbf{x}') = \frac{T_{ij}}{|T|} B_j(\mathbf{x}) \quad (41)$$

by equation (39).

We return to the Parker magnetostatic problem to consider the field $\mathbf{B}(\mathbf{x})$ deformed from the uniform field

$$\mathbf{B}'(\mathbf{x}') = (0, 0, B_0) \quad (42)$$

in \mathbf{x}' -space. Equation (41) gives the linear algebraic equations

$$T_{1j} B_j(\mathbf{x}) = \mathbf{B} \cdot \nabla x'_1 = 0, \quad (43)$$

$$T_{2j} B_j(\mathbf{x}) = \mathbf{B} \cdot \nabla x'_2 = 0, \quad (44)$$

$$T_{3j} B_j(\mathbf{x}) = \mathbf{B} \cdot \nabla x'_3 = |T| B_0, \quad (45)$$

to determine the three components of a non-equilibrium field $\mathbf{B}(\mathbf{x})$, that we have denoted as $\mathbf{B}_{neq}(\mathbf{x})$, now expressed in terms of fluid displacement $\mathbf{x}'(\mathbf{x})$ from the uniform field $\mathbf{B}'(\mathbf{x}') = (0, 0, B_0)$.

In \mathbf{x}' -space, the uniform field $\mathbf{B}'(\mathbf{x}') = (0, 0, B_0)$ has the Euler-potential representation:

$$[U'(\mathbf{x}'), V'(\mathbf{x}')] = \sqrt{B_0} [x'_1, x'_2] \equiv \sqrt{B_0} [x', y'], \quad (46)$$

$$\begin{aligned} \mathbf{B}'(\mathbf{x}') &= \nabla' U' \times \nabla' V' \\ &= B_0 \nabla' x'_1 \times \nabla' x'_2 \\ &= B_0 \hat{\mathbf{x}}' \times \hat{\mathbf{y}}' \\ &= B_0 \hat{\mathbf{z}}', \end{aligned} \quad (47)$$

inserting $(x'_1, x'_2, x'_3) \equiv (x', y', z')$ to be used alternatively for easy reading of equations written in terms of the Cartesian coordinates used in the Parker magnetostatic problem. The field

lines of $\mathbf{B}'(\mathbf{x}')$ are straight lines at right-angle to boundaries $z' = 0, L$, each a line-intersection between two perpendicular, flat Euler flux-surfaces of constant $U' = \sqrt{B_0}x', V' = \sqrt{B_0}y'$.

The Lagrangian transform (35) takes these planar Euler-potential surfaces in \mathbf{x}' -space into two families of non-planar surfaces of constant (U, V) in \mathbf{x} -space given by

$$\begin{aligned} [U(\mathbf{x}), V(\mathbf{x})] &= [U'(\mathbf{x}'), V'(\mathbf{x}')] \\ &= \sqrt{B_0} [x'_1(x, y, z), x'_2(x, y, z)] \\ &\equiv \sqrt{B_0} [x'(x, y, z), y'(x, y, z)]. \end{aligned} \quad (48)$$

For any given tangential velocity $\mathbf{v}'(x', y', 0)$ on $z' = 0$ producing $\mathbf{B}(\mathbf{x}) = \mathbf{B}_{neq}(\mathbf{x})$ from $\mathbf{B}'(\mathbf{x}')$, induction equations (25) and (26) determine the displaced boundary values of the Euler potentials of $\mathbf{B}(\mathbf{x})$ henceforth denoted by $[U_{bc}(x, y), V_{bc}(x, y)] = [U(\mathbf{x}), V(\mathbf{x})]_{z=0}$ in \mathbf{x} -space. Taking $\mathbf{v}'(x', y', 0)$ to be incompressible, $B_z = B_0$ remains unchanged on $z = 0$, which ensures that (U_{bc}, V_{bc}) is a solution of the PDE defined on $z = 0$

$$\begin{aligned} \frac{\partial(U_{bc}, V_{bc})}{\partial(x, y)} &= B_z(x, y, 0) \\ &= B_0, \end{aligned} \quad (49)$$

using equation (15).

Let us obtain the continuum of (U_{bc}, V_{bc}) produced by all the foot-point displacements that are spatially localized on boundary $z' = 0$. Therefore, we impose boundary condition

$$(x^2 + y^2)^{1/2} \rightarrow \infty, \quad [U_{bc}(\mathbf{x}), V_{bc}(\mathbf{x})] \rightarrow \sqrt{B_0} [x, y], \quad (50)$$

to solve PDE (49) for a fixed B_0 . Each of the continuum of solutions so obtained describes a distribution of the displaced foot-points realized on $z = 0$. In each foot-point configuration belonging to a non-equilibrium field $\mathbf{B}(\mathbf{x}) = \mathbf{B}_{neq}(\mathbf{x})$, its (displaced) foot-points are identified by the constant values of their Euler potentials $[U_{bc}(x, y), V_{bc}(x, y)]$. For each foot-point configuration realized on $z = 0$, PDEs (17) and (18) subject to boundary conditions

$$\text{at } z = 0, \quad (U, V) = [U_{bc}, V_{bc}], \quad (51)$$

$$\text{at } z = L, \quad (U, V) = \sqrt{B_0} [x, y]. \quad (52)$$

determine the Euler potentials (U, V) for a force-free field that is topologically equivalent to $\mathbf{B}_{neq}(\mathbf{x})$. We now identify $(U, V) \equiv \sqrt{B_0} [x'(\mathbf{x}), y'(\mathbf{x})]$ under equation (48). Therefore, the force-free boundary value problem (BVP) may be restated as an equivalent problem for the two components $[x'(\mathbf{x}), y'(\mathbf{x})]$ of a Lagrangian transform taking the uniform field $\mathbf{B}'(\mathbf{x}') = (0, 0, B_0)$ into a topologically available force-free field.

The nonlinearity of PDEs (17) and (18) pose two formidable issues for their BVP, either in terms of (U, V) or, equivalently, in terms of $[x'(\mathbf{x}), y'(\mathbf{x})]$. The first issue is that the BVP may not have a solution in the sense of mathematical analysis and, if a solution exists, it may not be unique; see the BVPs of the Sturm-Liouville theory of linear PDEs in Conversation V. The BVP has a clear physical meaning, so it must have a physically meaningful solution. With this clarification, if no analytical solution to the BVP can be found in a particular case, a discontinuous solution is then the one to be constructed in context of the physical problem. The discontinuity in the force-free field allowed by physics is not arbitrary. The equilibrium field may embed tangential discontinuities (TDs), each TD carrying a discrete current sheet (CS) subject to two equilibrium requirements. Each TD must be in force balance with the surrounding equilibrium field. Integrating the Lorentz force over each TDs yields the requirement that the magnetic pressure B^2 is continuous across the TD. The second requirement is for the current density in the continuous parts of the field and the discrete CSs in all the TDs to give a global field under Ampere's law that is topologically equivalent to the fields $\mathbf{B}_{neq} \in \mathcal{S}[\tau]$. In mathematical analysis, the force-free fields with TDs are given the name of weak solutions in which the TDs satisfy the force-free equation (2) in the integrated sense. In physics, TDs are an essential property of the perfect fluid conductors, just as the hydrodynamic shock is an essential property of compressive inviscid fluid.

The distinction between a solution to the BVP posed by PDEs (17) and (18) and the physics of the Parker magnetostatic problem is far reaching beyond the first issue discussed above. The Parker problem demands specifically for a force-free field topologically equivalent to the fields $\mathbf{B}_{neq} \in \mathcal{S}[\tau]$. PDEs (17) and (18) imposes the force-free condition (2) on (U, V) at each point in space, a condition independent of the topology of the field. Topology is a global property not lodged at any single point in space. When a solution to the BVP exists, the topology of the field obtained is then not one prescribed *a priori* but one that accommodates the force-free condition under the implicit assumption of analyticity. Since each field line is a curve of constant (U, V) , boundary conditions (51) do impose a particular aspect of the field topology, namely the foot-point connectivity shared by all the continuous fields $\mathbf{B}_{neq} \in \mathcal{S}[\tau]$. However, that the field lines of constant (U, V) must possess this global foot-point connectivity is only an implication of the boundary conditions (51), not explicitly embodied in the governing PDEs (17) and (18), leaving open to several possibilities, depending on the boundary distributions $[U_{bc}, V_{bc}]$.

One possibility is that an analytic solution, in fact, exists for a particular pair $[U_{bc}, V_{bc}]$ in which case, the implied foot-point connectivity is naturally realized. Provided the field solution with the implied foot-point connectivity also has all the other topological properties of $\mathbf{B}_{neq} \in \mathcal{S}[\tau]$, we do not have a solution of the Parker problem. As an illustration, the (U, V)

solution to the BVP may contain spatially localized maxima or minima in U or V within the fluid domain, expected of elliptic PDEs. The solution thus contains closed magnetic flux surfaces of constant U or V , in addition to the main field whose field lines are anchored to $z = 0, L$. Therefore, we do not have a solution to the Parker problem. It might be said that the extraneous bubbles of closed magnetic flux, each not attached to the boundaries, are self-consistently included by mathematics to meet the simultaneous requirements of the force-free condition and the imposed foot-point connectivity. Without the extraneous bubbles, the two requirements cannot both be met, a conclusion predicated by the assumption of analyticity. If the fields and fluids in the bubbles are artificially removed, the surrounding fluid would collapse into the evacuated space, forming TDs in a global, all-anchored, force-free field as a true solution to the Parker problem. This discussion leads us to the direct proof by the Parker optical analogy that the loss of analyticity is a generally expected, physical property of force-free fields; E. N. Parker, *The optical analogy for vector fields*. Phys. Fluids B 3, 2652.

Consider a solution (U, V) to the BVP, defining a force-free magnetic field \mathbf{B}_{ff} in terms of its the flux surfaces of constant (U, V) , the field assumed for the present to be analytical. The field \mathbf{B}_{ff} and its current density $\nabla \times \mathbf{B}_{\text{ff}}$ are parallel by force-free equation (2), so the flux surfaces of the field and those of the current density coincide. The fact that the current density has no component perpendicular to the flux surfaces of constant (U, V) , i.e.,

$$(\nabla \times \mathbf{B}_{\text{ff}}) \cdot \nabla U = (\nabla \times \mathbf{B}_{\text{ff}}) \cdot \nabla V \equiv 0, \quad (53)$$

implies that \mathbf{B}_{ff} is a potential field,

$$\begin{aligned} \mathbf{B}_{\text{ff}} &= \nabla \phi(\xi_1, \xi_2), \\ &= \left(-\frac{1}{h_1} \frac{\partial \phi}{\partial \xi_1}, -\frac{1}{h_2} \frac{\partial \phi}{\partial \xi_2} \right). \end{aligned} \quad (54)$$

in each 2D, non-Euclidean flux surface described with orthogonal coordinates (ξ_1, ξ_2) , Lamé coefficients $[h_1(\xi_1, \xi_2), h_2(\xi_1, \xi_2)]$ defining the square of the path-length element: $ds^2 = h_1^2 d\xi_1^2 + h_2^2 d\xi_2^2$, and the gradient operator defined in the (ξ_1, ξ_2) -surface. On this flux surface, the field lines are described by the ODEs (1) that take the form

$$\begin{aligned} Bh_1 \frac{d\xi_1}{ds} &= -\frac{1}{h_1} \frac{\partial \phi}{\partial \xi_1}, \\ Bh_2 \frac{d\xi_2}{ds} &= -\frac{1}{h_2} \frac{\partial \phi}{\partial \xi_2}, \end{aligned} \quad (55)$$

where $B(\xi_1, \xi_2)$ is the field amplitude. By definition of ds , the field lines on the flux surface is given by the first order PDE

$$[\nabla \phi(\xi_1, \xi_2)]^2 = \left(\frac{1}{h_1} \frac{\partial \phi}{\partial \xi_1} \right)^2 + \left(\frac{1}{h_2} \frac{\partial \phi}{\partial \xi_2} \right)^2 = 0, \quad (56)$$

recognizable as the equation for optical ray paths in a medium with $\phi(\xi_1, \xi_2)$ and B identified respectively with the optical eikonal and index of refraction.

To relate this optical analogy to the mechanics of a force-free field, rewrite equation (2) as

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = (\mathbf{B} \cdot \nabla) \mathbf{B} - \nabla \left(\frac{1}{2} B^2 \right) = 0, \quad (57)$$

showing that the Lorentz force vanishes by a balance between the magnetic tension force and the isotropic magnetic pressure force. The force-free field has a heterogenous magnetic pressure which must be balanced by inward tension forces exerted by field lines in outwardly convex flux-surfaces. Under the frozen-in condition, the field lines can slip along the convex flux surfaces if not restrained by a similar force-balance along the surface. The geometry of the flux surfaces is thus essential for the global equilibrium, one effective geometric constraint being the mathematical imposition of an ignorable coordinate.

Returning to the force-free field \mathbf{B}_{ff} , its flux surfaces of constant (U, V) may be viewed to have started off as plane surfaces of the uniform straight field lines in the state $\mathbf{B}'(\mathbf{x}') = B_0 \hat{\mathbf{z}}'$ which get deformed arbitrarily under the Lagrangian transform $\mathbf{x}' = \mathbf{x}'(\mathbf{x})$ into a field \mathbf{B} given by algebraic equations (43) - (45), whereupon the demand is imposed on $\mathbf{x}' = \mathbf{x}'(\mathbf{x})$ to yield $\mathbf{B} = \mathbf{B}_{\text{ff}}$. Under these continuous deformations, each flux surface of constant (U, V) is continuous with field lines covering the surface, but care must be exercise to presume this remains the case when $\mathbf{B} = \mathbf{B}_{\text{ff}}$.

Under the force-free condition, the field lines on any flux surface is governed by the eikonal PDF (56) which describes the phenomenon of optical exclusion. The bending of light-rays away from regions of high refractive index, the lensing effect, can be so severe that regions of extremely high index of refraction is devoid of light rays entering such regions. In other words, approaching a balance of the magnetic tension force with the magnetic pressure force, the field lines in a given flux surface may be displaced aside and cease to cover the a local area of the surface, creating gaps or holes where none existed before. Through those holes, magnetic fluxes external to the given flux surface may then come together to form TDs across which B^2 is continuous. Thus, the assumption of an analytical \mathbf{B}_{ff} breaks down. That is, the imposition of the specific topology in $\mathcal{S}[\tau]$ is generally extraneous to the force-free PDEs (17) and (18), giving rise to an over-determining problem under the assumption of analyticity.

4. Concluding remarks

Special cases of the (U, V) BVP may have continuous field solutions that agree with the topology of a given $\mathbf{B}_{neq} \in \mathcal{S}[\tau]$, but more generally, the solutions of the (U, V) BVP must be extended to the weak solutions with TDs for the Parker problem. The proof by the optical analogy in context of the (U, V) BVP is general. The nature of the proof suggests that if the continuum set of continuous field solutions to the Parker problem is compared to those solutions with inevitable TDs, the former is a subset of their union of measure zero.

5. Coda: Parker Magnetostatic Theorem

We construct an understanding of the Parker Magnetostatic Theorem, hereafter the Theorem, in terms of the two-plate problem in its relationship with the boundary value problem (BVP) for a force-free field. The two problems are not precisely the same, clearly seen in the (U, V) representation (4) for a solenoidal field. The construction is explicit in the physical meanings of concepts and issues involved and in making several inferences about the generally intractable BVP for a 3D field. New in this coda is a general derivation of the Parker eikonal ϕ .

5.1. Statement of the Two-plate problem

The two-plate problem demands for a force-free field \mathbf{B}_{ff} in $0 < z < L$ subject to (i) the boundary condition $B_z = B_0$ at $z = 0, L$ and (ii) the field having an arbitrarily prescribed field topology of a specific kind, hereafter denoted by \mathcal{T}_{ff} . That is, the force-free field is continuously deformed under the induction equation from the uniform field $\mathbf{B}_0 = B_0 \hat{\mathbf{z}}$, subject to two boundary conditions. On the boundary $z = 0$, rigidity is first relaxed and then restored after imposing an arbitrarily prescribed, tangential, incompressible, continuous displacement of the foot-points (fpts) of \mathbf{B}_0 , this boundary displacement hereafter denoted by $\mathcal{D}_{z=0}$. Whereas, on $z = L$, the fpts suffer no displacement. The boundary displacement $\mathcal{D}_{z=0}$ is smoothly continued into $0 < z < L$ to zero displacement at $z = L$.

5.2. Fundamentals

5.2.1. Perfectly conducting fluid, field topology and the frozen-in condition

Induction equation (20) is often taken out of context, useful to clarify for our purpose here. Particles and parcels of fluid have physical identities whereas magnetic fields by themselves have no identities. In a perfect conductor, described by the induction equation, magnetic flux is conserved over every fluid surface as the latter moves and deforms with velocity \mathbf{v} . It is this frozen-in property that renders each flux surface a permanent identity. That is, any two flux surfaces at different times are the same physical object by the same fluid on the surfaces. Field lines as intersections of flux surfaces are thus also physically identifiable objects by the same fluid on each line.

In the two-plate problem, if $B_z \neq 0$ on rigid boundaries $z = 0, L$, then $\mathbf{v} = 0$ for all time on these boundaries by the condition that an electric field must be tangentially continuous

across any fluid surface. In this case, the distributions of B_z on $z = 0, L$ are constant in time.

The uniform field \mathbf{B}_0 embedded in the fluid may be deformed continuously into a non-equilibrium field bearing a nonzero Lorentz force, from which a reversed deformation trivially returns the field back to \mathbf{B}_0 . By its textbook definition, the topology $\mathcal{T}_{\text{unif}}$ of the flux surfaces and field lines, as geometric objects, comprise those properties that are invariant under all continuous deformations. For example, with $\mathbf{v} = 0$ at the rigid boundaries, the field-line connectivity between the two boundaries is invariant as the field is deformed. From boundary to boundary, a field line winding in one sense around a second field line must also wind back in the opposite sense to be consistent with no change of locations in the first field line's fpts. A useful operational meaning of $\mathcal{T}_{\text{unif}}$ of the anchored field \mathbf{B}_0 is to say that the continuum of all the non-equilibrium fields deformable from \mathbf{B}_0 are topologically equivalent under the induction equation, all the fields continuously deformable one into another, and, all the fields deformable back to \mathbf{B}_0 .

The above topological properties take a non-trivial significance when we relax the rigidity of $z = 0$ to impose the fpt displacement $\mathcal{D}_{z=0}$ corresponding to $\mathbf{v} \neq 0$ on $z = 0$. Subsequently restoring the rigidity of $z = 0$ locks the deformed field \mathbf{B}_{neq} in $0 < z < L$ with a topology \mathcal{T}_{neq} generally not the same as $\mathcal{T}_{\text{unif}}$. That is, \mathbf{B}_{neq} cannot be further deformed back to the uniform field \mathbf{B}_0 except by again relaxing rigidity of $z = 0$ followed by a reversal of the fpt displacement $\mathcal{D}_{z=0}$ on that boundary. The two-plate problem seeks a force-free field \mathbf{B}_{ff} with the same topology \mathcal{T}_{neq} into which \mathbf{B}_{neq} may be deformed under the induction equation and the rigid boundary conditions.

5.2.2. *Tangential discontinuities (TDs) and their current sheets (CSs)*

A field with TDs in a perfect conductor is physically and fundamentally admissible. TDs in a force-free field are not arbitrarily but must satisfy force-free equation (2) in the integrated sense, the magnetic pressure $|\mathbf{B}_{\text{ff}}|^2$ required to be continuous across each TD. The discrete CSs in the TDs are described by the Dirac δ -function. The global field is then described by Ampere's law in terms of these CSs and the the continuous part of the force-free current density. The construction of such a TD-bearing force-free (TDFF) field is one of treating the TDs as free boundaries to be determined as an essential part of the BVP for the continuous part of the field, analogous to the soap-bubble problem.

Any TDFF field in a perfectly conducting fluid may be deformed under the induction equation into an everywhere continuous non-equilibrium field \mathbf{B}_{neq} , but the the deformation

clearly cannot be everywhere continuous in order to remove the field discontinuities. Such a not-everywhere discontinuous deformation is subject to the induction equation requiring conservation of magnetic flux across every fluid surface. All such \mathbf{B}_{neq} created in a fluid with rigid boundary share a common topology \mathcal{T}_{neq} which is, by definition, also the topology of the TDFF field.

In the two-plate problem, we have limited ourselves to continuous fluid deformations, by which the textbook definition of topology follows. Flux conservation on every fluid surface is *the* fundamental property of the induction equation, a property that must also apply in a discontinuous fluid displacement. The continuous deformation of a continuous field yields another continuous field. Thus, further deforming the \mathbf{B}_{neq} created in $0 < z < L$ by strictly continuous displacements can never produce a TDFF field, even if such a TDFF exists with the same topology \mathcal{T}_{neq} . By the same consideration, if a continuous force-free field exists with the same topology \mathcal{T}_{neq} , then it is available to \mathbf{B}_{neq} by continuous displacements. Here and elsewhere, we are dealing with deformations in fluids bounded by rigid walls, unless stated otherwise.

There are, of course, everywhere-continuous force-free fields but we must accept that there are also the TDFF fields. The Theorem makes the fundamental point that TDFF fields are the only ones available to \mathbf{B}_{neq} for *most* prescribed 3D \mathcal{T}_{neq} , which is what we set out to explain in this Coda. The essential point of the explanation is that, in the full freedom of 3D motions, a topologically complex 3D field generally achieves its force-free state under the induction equation by having two or more flux systems pushing other flux systems completely out of their ways, enabling the in-pushing systems to make contact at TD-surfaces.

Two points are worthy of address at this juncture, the breakdown of the frozen-in condition and the dynamical meaning of achieving the force-free state.

5.2.3. *Near-ideal versus strictly ideal fluids*

Perfect conductivity is a physical idealization, the idealization never to mean that its physics is unreal. Let us deal with a fluid of resistivity $\eta = \frac{c^2}{4\pi\sigma}$ in cgs units where c is the speed of light and σ the conductivity, taking η and σ as given constants. The time scale of the resistive diffusion of a field by current dissipation is of the order of $\tau_\eta = \frac{l^2}{\eta}$, where l is a characteristic length scale of the field, to be compared to the time scale $\tau_{v_0} = \frac{l}{v_0}$ of a macroscopic velocity v_0 on the same length scale. The dimensionless magnetic Reynolds number $R_m = \frac{\tau_\eta}{\tau_{v_0}} = \frac{lv_0}{\eta}$ is orders of magnitude large in many astrophysical systems. In such

systems, a macroscopic field behaves as in a perfect conductor until nonlinear MHD develops sufficiently small scales in a $R_m \approx 1$ locality where the frozen-in condition breaks down for resistive current dissipation.

The small scales develop as fast as they are removed resistively by magnetic reconnection. How fast this resistive process can take place is a major physical problem interesting in its own right, but the point is clear that magnetic reconnection must be intermittent in space and time. As soon as the small scales are resistively removed, the $R_m \gg 1$ condition is restored. The Theorem identifies the origin of such an ever pervasive developments of $R_m \rightarrow 0$ condition in 3D complex field, by the spontaneous formation of TDs as a $R_m \gg 1$ near-ideal process in complex fields.

The Spitzer temperature-dependent resistivity gives a magnetic Reynolds number $R_m \geq 10^8$ for most observable phenomena in the solar corona, R_m getting to order unity only when l is of the order of 100 km, the (electron) mean-free path of the million-degree highly conducting corona. In other words, the macroscopic coronal fields evolve effectively as in a perfect fluid conductor with a propensity to form TDs as a high-conductivity process. TD formation does not reach a mathematical discontinuity in the field, always overtaken by a simultaneous breakdown of the frozen-in condition as well as the fluid description. The resistivity triggered is thus expected to involve the complex plasma-kinetic processes outside the fluid description.

5.2.4. *Magnetic energy and the frozen-in condition*

The force-free field is a simple form of force equilibrium, in a fluid so tenuous that all other forms of energy are negligible except for the magnetic energy \mathcal{E} . We can imagine this fluid to be viscous so that in an actual mechanical evolution, viscosity irreversibly removes and keeps the kinetic energy small. The unbounded domain $0 < z < L$ allows a loss of kinetic energy via hydromagnetic waves escaping to infinity in the horizontal x and y directions. Thus, the equilibrium of \mathbf{B}_{neq} , in a force-free state, is one of minimum magnetic energy, $\mathcal{E} = \mathcal{E}_{\text{min}}$, not any force-free state but one that possesses a specific topology \mathcal{T}_{neq} permanently preserved under the frozen-in condition.

This $\delta\mathcal{E} = 0$ state is an extremum over all the *continuous* \mathbf{B}_{neq} admissible in $0 < z < L$ with rigid boundaries. Yet, this state must generally contain TDs, according to the Theorem. That the solution to the variational problem is discontinuous despite the extremization being taken over continuous fields is a general but often overlooked feature of variational calculus (Courant & Hilbert 1963). That the magnetic energies \mathcal{E} of the topologically equivalent

fields have a largest lower bound \mathcal{E}_{\min} is not being questioned. Whereas, this bound does not require the actual existence of a *continuous* field that realizes its value \mathcal{E}_{\min} exactly. When no analytical force-free state exists for \mathbf{B}_{neq} with its prescribed \mathcal{T}_{neq} , continuous deformation of \mathbf{B}_{neq} can bring the field to approximate a TDFF field of the same topology, the only available force-free state with \mathcal{T}_{neq} , as close as desired in the limit $\mathcal{E} \rightarrow \mathcal{E}_{\min}$. This nature of variational calculus and the TDs as a fundamental property of the perfect fluid conductor prepare us for understanding the Theorem.

To again emphasize a point, TDs and their CSs are admissible physical properties of perfect conductivity. The Theorem does contain an intellectual surprise, not in the existence of TDFF but in the fact that TDFFs are the only equilibria for *most* prescribed \mathcal{T}_{neq} .

5.2.5. Volume-filling field lines

A 3D-varying field disconnected from the boundary may contain infinitely-long, volume-filling, ergodic field-lines. An ergodic line in 3D space as a unique solution to the field-line ODEs (1) is well defined along its infinite length but its geometric interpretation as the intersection of a pair of flux surfaces in this case is meaningful only locally. Such a geometric trouble does not arise for the two-plate problem seeking a force-free field stipulated to possess *only* anchored field lines. The field's (U, V) flux surfaces are global and identifiable for all time, each flux surface bodily carried along by the fluid on it as described by ideal induction equations (25) and (26).

A useful geometric clarification. Given a field with its uniquely defined field lines, the geometric choice of the (U, V) -families of flux surfaces containing the field lines is not unique. For each choice of (U, V) -pair, all other flux surfaces of the given field are each a surface of constant $F(U, V)$, F being a suitably defined function of two variables. In other words, a pair of independent functions of two variables $[(F(U, V), G(U, V))]$ defines another ordering of the same field lines into two independent families of flux surfaces of constant (F, G) as functions of space. Physical properties are, of course, independent of the free choice of (U, V) -representation, a point to remind ourselves with later in our discussion.

With the fundamentals laid out in this subsection, we proceed to a (U, V) formulation of the two-plate problem.

5.3. Flux-function description of the two-plate problem

The (U, V) representation is naturally suited for formulating the two-plate problem. With its (U, V) flux surfaces globally defined, a field evolves in a conceptually straightforward manner under induction equations (25) and (26). The velocity \mathbf{v} simply transports U and V as fixed constants on each fluid particle. Introduce the continuous Lagrangian transform of fluid locations, $\{\mathcal{L}(\mathbf{v}) : \mathbf{x}' = (x', y', z') \rightarrow \mathbf{x} = (x, y, z)\}$, describing each fluid particle having moved from \mathbf{x}' to \mathbf{x} in a prescribed \mathbf{v} . The deformation is *reversible*, so the function $\mathbf{x}(\mathbf{x}')$ and its inverse $\mathbf{x}'(\mathbf{x})$ are well defined. The continuous deformation of a field from $\mathbf{B}'(\mathbf{x}')$ to $\mathbf{B}(\mathbf{x})$ by \mathbf{v} is then equivalent to the transformation,

$$[\mathbf{B}'(\mathbf{x}'), \mathbf{B}(\mathbf{x})] = [\nabla' U'(\mathbf{x}') \times \nabla' V'(\mathbf{x}'), \nabla U(\mathbf{x}) \times \nabla V(\mathbf{x})], \quad (58)$$

$$[U'(\mathbf{x}'), V'(\mathbf{x}')] = [U(\mathbf{x}), V(\mathbf{x})], \quad (59)$$

where ∇' is the operator in \mathbf{x}' coordinate-space. In equation (59), (U', V') and (U, V) as explicit functions of physical space define the reversible Lagrangian transform $\mathbf{x}' \leftrightarrow \mathbf{x}$ connecting these two states of the field .

5.3.1. Setting up \mathbf{B}_0 , \mathbf{B}_{neq} and \mathbf{B}_{ff}

The uniform field is given by

$$(U_{\text{unif}}, V_{\text{unif}}) = (\sqrt{B_0}x, \sqrt{B_0}y), \quad (60)$$

$$\begin{aligned} \mathbf{B}_0 &= \nabla U_{\text{unif}} \times \nabla V_{\text{unif}} \\ &= B_0 \hat{\mathbf{x}} \times \hat{\mathbf{y}} \\ &= B_0 \hat{\mathbf{z}}. \end{aligned} \quad (61)$$

Its flux surfaces of constant $(U_{\text{unif}}, V_{\text{unif}})$ are respectively planes of constant x and constant y , the straight field lines of $\mathbf{B}_0 = B_0 \hat{\mathbf{z}}$ uniformly distributed on each plane. Subject to continuous fluid motions these unbounded planar flux-surfaces are deformed into homologous, boundless, warped surfaces, each continuously covered with field lines deformed from the straight field-lines of the uniform field.

To create \mathbf{B}_{neq} , consider the continuous deformation taking each fluid particle at $\mathbf{x}' = (x', y', z')$ to $\mathbf{x}'' = (x'', y'', z'')$,

$$(x'', y'', z'') = \left[\left(1 - \frac{z'}{L}\right) f(x', y'), \left(1 - \frac{z'}{L}\right) g(x', y'), z' \right], \quad (62)$$

a strictly-horizontal displacement at each height z' , freely prescribed by pair of functions $D \equiv [f(x', y'), g(x', y')]$. Let us spatially localize the displacement on the $z' = 0$ plane, i.e., $(f, g) \rightarrow (x', y')$ as $R = \sqrt{x'^2 + y'^2} \rightarrow \infty$. Equation (62) describes a global displacement scaled uniformly with height z' , from the displacement $\{\mathcal{D}_{z=0} : (x'', y'', 0) = [f(x', y'), g(x', y'), 0]\}$ at boundary $z' = 0$ to zero displacement $(x'', y'', L) = (x', y', L)$ at boundary $z = L$. The uniform field \mathbf{B}_0 in \mathbf{x}' -space is deformed by this displacement into the non-equilibrium field \mathbf{B}_{neq} in \mathbf{x}'' -space expressed by

$$\mathbf{B}_{\text{neq}}(\mathbf{x}'') = \nabla'' U_{\text{neq}}(\mathbf{x}'') \times \nabla'' V_{\text{neq}}(\mathbf{x}''), \quad (63)$$

$$[U_{\text{neq}}(\mathbf{x}''), V_{\text{neq}}(\mathbf{x}'')] = [U_{\text{unif}}(\mathbf{x}'), V_{\text{unif}}(\mathbf{x}')]. \quad (64)$$

A property of the above fluid displacement needs to be explained. Strictly-horizontal 3D fluid displacements, of which equation (62) is an example, are sufficient to carry out all admissible deformations of \mathbf{B}_0 . In induction equation (20), an arbitrary component of the velocity everywhere parallel to the field has no inductive influence on the field. In other words, superposing an arbitrary field-aligned component on a deforming fluid displacement produces the same field deformation. This free component permits the assumption of a horizontal fluid displacement at all height z' in equation (62), with no loss of generality.

The fpt displacement $\{\mathcal{D}_{z'=0} : (x'', y'', 0) = [f(x', y'), g(x', y'), 0]\}$ on $z' = 0$ is assumed to be incompressible, so $\mathbf{B}_{\text{neq}}(\mathbf{x}'')$ satisfies the boundary condition $B_{\text{neq}, z''} = B_0$ at $z'' = 0$ in its Cartesian \mathbf{x}'' -space. This boundary condition poses the PDE on the plane $z'' = 0$,

$$\begin{aligned} B_{\text{neq}, z''} &= \frac{\partial [U_{\text{neq}}(x'', y'', 0), V_{\text{neq}}(x'', y'', 0)]}{\partial [x'', y'']} \\ &\equiv \left[\frac{\partial U_{\text{neq}}}{\partial x''} \frac{\partial V_{\text{neq}}}{\partial y''} - \frac{\partial U_{\text{neq}}}{\partial y''} \frac{\partial V_{\text{neq}}}{\partial x''} \right]_{z''=0} \\ &= B_0, \end{aligned} \quad (65)$$

subject to the far boundary conditions $[U_{\text{neq}}(x'', y'', 0), V_{\text{neq}}(x'', y'', 0)] \rightarrow [x'', y'']$ as $\sqrt{x''^2 + y''^2} \rightarrow \infty$, expressing the localization of the fpt displacement $\mathcal{D}_{z'=0}$. Each solution of PDE (65), denoted by $[U^*(x'', y''), V^*(x'', y'')]$, is the boundary value at $z'' = 0$ of the flux functions $[U_{\text{neq}}(\mathbf{x}''), V_{\text{neq}}(\mathbf{x}'')] of $\mathbf{B}_{\text{neq}}(\mathbf{x}'')$ consistent with the two-plate boundary condition $B_{\text{neq}, z''} = B_0$ at $z'' = 0$.$

As a separate consideration, the solution $[U^*(x'', y''), V^*(x'', y'')]$ is the result of deforming the boundary flux functions $[U_{\text{unif}}(x', y', 0), V_{\text{unif}}(x', y', 0)]$ of \mathbf{B}_0 by a specific fpt displacement $\mathcal{D}_{z'=0}$ under induction equations (25) and (26). Therefore, we have the relationship

$$[U^*(x'', y''), V^*(x'', y'')] = [U_{\text{unif}}(x', y', 0), V_{\text{unif}}(x', y', 0)], \quad (66)$$

interpretable in two ways. We may say, the left hand side of equation (66) is the result of a prescribed fpt displacement $\mathcal{D}_{z'=0}$ deforming the boundary flux-functions of \mathbf{B}_0 on $z' = 0$. Alternatively, given $[U^*(x'', y''), V^*(x'', y'')]$, equation (66) identifies the specific fpt displacement $\mathcal{D}_{z'=0}$ relating the two sets of boundary values.

Denote by $\mathcal{M}(U^*, V^*)$ the manifold of solutions to PDE (65), each solution the result of a specific incompressible fpt displacements $\mathcal{D}_{z'=0}$. Therefore, we may identify each non-equilibrium field $\mathbf{B}_{\text{neq}}(\mathbf{x}'')$ created in the two-plate problem by its boundary values $(U^*, V^*) \in \mathcal{M}$ at $z'' = 0$ in the triple relationship:

$$[U_{\text{neq}}(x'', y'', 0), V_{\text{neq}}(x'', y'', 0)] = [U^*(x'', y''), V^*(x'', y'')] = [U_{\text{unif}}(x', y', 0), V_{\text{unif}}((x', y', 0))], \quad (67)$$

the first equality as a statement of the boundary condition on $[U_{\text{neq}}(x'', y'', 0), V_{\text{neq}}(x'', y'', 0)]$ at $z'' = 0$ and the second equality as the specification of the corresponding $\mathcal{D}_{z'=0}$. In fact, the flux-functions of \mathbf{B}_0 given by equation (60) substituted into the second equality describes $\mathcal{D}_{z'=0}$ as an explicit function $\mathbf{x}' = \mathbf{x}'(\mathbf{x}'')$ on the lower boundary:

$$z' = z'' = 0, \quad \sqrt{B_0} [x', y', 0] = [U^*(x'', y''), V^*(x'', y'')]. \quad (68)$$

To summarize, starting with a choice of a pair of boundary values $(U^*, V^*) \in \mathcal{M}$, we are looking at a non-equilibrium field $\mathbf{B}_{\text{neq}}(\mathbf{x}'')$, given by equations (58) and (59), which is deformed from \mathbf{B}_0 by fluid displacement (62).

Having so created $\mathbf{B}_{\text{neq}}(\mathbf{x}'')$ and restored rigidity at both boundaries $z'' = 0, L$, this non-equilibrium field can be further deformed by a whole continuum of continuous fluid displacements in $0 < z < L$ into other non-equilibrium fields all topologically equivalent and sharing the same boundary values $(U^*, V^*) \in \mathcal{M}$. In other words, we have constructed a manifold $\mathcal{B}_{\mathcal{T}}$ of continuous fields, each field $\mathbf{B}''(\mathbf{x}'')$ with flux functions $[U''(\mathbf{x}''), V''(\mathbf{x}'')]$, continuously deformable into any other, and all the fields collectively defining a common topology denoted by \mathcal{T} .

An important point of clarification, the construction starts with the specific field $\mathbf{B}_{\text{neq}}(\mathbf{x}'')$ deformed from \mathbf{B}_0 by the specific deformation (62). This step defines topology \mathcal{T} by its physical realization in the particular field $\mathbf{B}_{\text{neq}}(\mathbf{x}'')$. Further continuous deformations of $\mathbf{B}_{\text{neq}}(\mathbf{x}'')$ produce the topologically equivalent, continuous fields that populate the manifold $\mathcal{B}_{\mathcal{T}}$. The physical meaning of \mathcal{T} is thus expressed by $\mathcal{B}_{\mathcal{T}}$ containing the continuum of all topologically equivalent fields each continuously deformable into any other under the induction equation subject to the rigid boundary conditions at $z'' = 0, L$. The construction of $\mathcal{B}_{\mathcal{T}}$ could have started with any of the topologically equivalent fields the manifold contains. The field $\mathbf{B}_{\text{neq}}(\mathbf{x}'')$ has no significance beyond its role in the construction as a *reference field* realizing topology \mathcal{T} .

We are now ready to pose the two-plate problem for a force-free field \mathbf{B}_{ff} .

We first give a compact description of $\mathcal{B}_{\mathcal{T}} [U^*(x, y), V^*(x, y)]$ as a manifold of continuous fields \mathbf{B} in \mathbf{x} -space sharing a topology \mathcal{T} and sharing a pair of boundary values $[U^*(x, y), V^*(x, y)] \in \mathcal{M}$ imposed on the flux functions of \mathbf{B} :

$$z = 0, \quad [U(x, y, 0), V(x, y, 0)] = [U^*(x, y), V^*(x, y)], \quad (69)$$

$$z = L, \quad [U(x, y, L), V(x, y, L)] = [x, y]. \quad (70)$$

The topology \mathcal{T} is explicitly expressed by the equation:

$$[U(x, y, z), V(x, y, z)] = [U(x'', y'', z'')_{\text{neq}}, V^*(x'', y'', z'')_{\text{neq}}], \quad (71)$$

relating the flux-functions (U, V) of each $\mathbf{B} \in \mathcal{B}_{\mathcal{T}} [U^*(x, y), V^*(x, y)]$ in \mathbf{x} -space to (U'', V'') defining the reference field \mathbf{B}_{neq} in \mathbf{x}'' -space. For each specific \mathbf{B} , equation (71) defines the Lagrangian fluid displacement $\mathbf{x} = \mathbf{x}(\mathbf{x}'')$ or its equivalent reversed displacement $\mathbf{x}'' = \mathbf{x}''(\mathbf{x})$ that transform $\mathbf{B}(\mathbf{x}) \leftrightarrow \mathbf{B}_{\text{neq}}(\mathbf{x}'')$. Worthy of emphasis is that both \mathbf{B} and \mathbf{B}_{neq} belong to $\mathcal{B}_{\mathcal{T}} [U^*(x, y), V^*(x, y)]$ and satisfy boundary conditions (69) and (70).

To complete the description of $\mathcal{B}_{\mathcal{T}} [U^*(x, y), V^*(x, y)]$, we cite the topological origin of \mathbf{B}_{neq} as the field deformed from the uniform field \mathbf{B}_0 under fluid displacement (62). This fluid displacement involves relaxing the rigidity of $z' = 0$ in \mathbf{x}' -space where \mathbf{B}_0 is continuously deformed into \mathbf{B}_{neq} in \mathbf{x}'' -space. Equation (62) gives the specifics of the Lagrangian displacement $\mathbf{x}'' = \mathbf{x}''(\mathbf{x}')$, in terms of which \mathbf{B}_{neq} is given by equations (63) and (64).

The two-plate problem thus seeks the special field $\mathbf{B}_{\text{ff}} \in \mathcal{B}_{\mathcal{T}} [U^*(x, y), V^*(x, y)]$ that satisfies the force-free equations (2) and (3). This statement of problem as well as the construction of manifold $\mathcal{B}_{\mathcal{T}} [U^*(x, y), V^*(x, y)]$ assume analyticity. Under these assumptions, if a force-free field, in fact, exists in $\mathcal{B}_{\mathcal{T}} [U^*(x, y), V^*(x, y)]$, then its flux functions (U, V) satisfy the coupled PDEs (11) and (12) subject to boundary conditions (69) and (70), this boundary value problem hereafter referred to as the BVP. In this case, we see that the BVP, as naturally posed here, says nothing about the constraint expressed by equation (71) relating \mathbf{B}_{ff} to \mathbf{B}_{neq} via the common topology \mathcal{T} of the two fields. In fact, this constraint is completely outside the definition of the BVP. Thus, under the assumptions of analyticity and existence of $\mathbf{B}_{\text{ff}} \in \mathcal{B}_{\mathcal{T}} [U^*(x, y), V^*(x, y)]$, the BVP may or may not satisfy constraint (71).

There is the added complications of the formidably nonlinear BVP as posed may have no solutions, and, when an analytical solution does exist, the solution may not be unique. Non-existence and non-uniqueness of solutions in this case are intrinsic properties of the BVP. Imposing an extraneous constraint, like equation (71), can only reduce but not increase the number of admissible (existing) solutions of the BVP.

Now, a force-free field as a $\delta\mathcal{E} = 0$ minimum energy state always exists by the physics of the two-plate problem, allowing for force-free fields with embedded equilibrium TDs. Here we recognize that the variational definition of the force-free state as $\delta\mathcal{E} = 0$ over *continuous fields* of the same topology \mathcal{T} is a general complete statement, whereas force-free equations (2) and (3) are limited by the assumption of an analytical field.

5.4. The analytical nature of the (U, V) -BVP

The BVP seeks a solution (U, V) of the PDEs (11) and (12),

$$\nabla \cdot [(\nabla U \cdot \nabla V) \nabla U - |\nabla U|^2 \nabla V] = 0, \quad (72)$$

$$\nabla \cdot [(\nabla U \cdot \nabla V) \nabla V - |\nabla V|^2 \nabla U] = 0, \quad (73)$$

reproduced here from Section 2, subject to boundary conditions (69) and (70). These boundary conditions impose a specific magnetic connectivity between the boundaries $z = 0, L$, by virtue of each field line of constant (U, V) being rooted at like values of (U, V) on the boundaries. A solution of the BVP, if it exists, will possess this imposed magnetic connectivity, of course.

The statement BVP is mathematically complete in the sense that if a solution exists it can be constructed within the structure of the BVP. Non-existence and uniqueness of solutions are meaningful in terms of that internal structure of the BVP. A continuum of other invariant topological properties are embodied in constraint (71), describing the 3D mutual and collective braidings and entanglements of the flux tubes common to the reference field \mathbf{B}_{neq} and the force-free field to be constructed for the two-plate problem. This extraneous constraint cannot be meaningfully coupled to the BVP because the PDEs (72) and (73) are, in a strict mathematical sense, unrelated to the force-free field that the two-plate problem seeks.

PDEs (72) and (73) determine its force-free field by a point-by-point imposition of the force-free condition, subject to boundary conditions (69) and (70). The extrapolation of this field from one boundary to the other shall have a flow of magnetic flux as required for this point-wise force-free condition.

If this extrapolation can be self-consistently successful, then an analytical solution to the BVP exists to be discovered. There may be several such solutions, allowing for non-uniqueness. Then, any one of these solutions is a true solution to the corresponding two-plate problem provided the true solution complies with constraint (71) for the topologically equivalent fields of a particular $\mathcal{B}_{\mathcal{T}} [U^*(x, y), V^*(x, y)]$. It is conceivable that none of the

available analytical solutions complies with constraint (71), in which case, the solution to the two-plate problem must be a TD-embedding force-free field to be constructed such that it is compatible with topology \mathcal{T} .

There is a non-trivially interesting possibility, easily shown to exist by direct construction of analytical solutions to a whole class of the BVP. These force-free field solutions contain bubbles of magnetic flux that are embedded in the global field anchored at $z = 0, L$. The anchored part of the field, of course, are automatically consistent with the magnetic connectivity between the two boundaries built into boundary conditions (69) and (70), but may have windings and entanglements of flux tubes inconsistent with constraint (71). In 3D fields, the solution of the BVP may contain sub-volumes of flux, with or without volume-filling ergodic field lines. The distinct windings and entanglements in the BVP solution may then be accounted for by the anchored flux making their ways from boundary to boundary, avoiding the detached fluxes arising from the point-wise force-free condition applied to the global field. This solution of the BVP is, of course, not a true solution of the two-plate problem by the fact that the latter must have no inclusion of flux systems detached from the boundaries.

Now, given an analytical solution of the BVP with detached systems of fluxes, one may artificially remove all the detached fluxes and let the altered field relax to an energy-minimum $\delta\mathcal{E} = 0$ force-free state, in which case, the latter by reasonable consideration must contain TDs to preserve the invariant topology \mathcal{T} of the altered field.

The above analysis shows that the 3D BVP generally has no analytical solution for an arbitrarily prescribed \mathcal{T} . Once so recognized, the reason for the general absence of an analytical solution for the BVP is clear to see. Under the induction equation, flux tubes are topologically invariant bodies of fluid, and, in their evolution to a force-free state $\delta\mathcal{E} = 0$, spatially apart flux tubes can push other flux tubes out of their ways to come into contact, studied in rich analysis and demonstrations in the book *Spntaneous Current-Sheets in Magnetic Fields* by E. N. Parker (1994), Oxford University Press.

To summarize, the BVP is distinct from the two-plate problem in the following way. The BVP leaves the field topology of the solution it seeks quite unconstrained except for the magnetic connectivity between the two plates dictated by boundary conditions (69) and (70). The full topology of the solution field is then defined as required by the force-free condition at each point of space, assuming that the point-wise condition can be met by the presence of detached flux system self-consistently embedded in the global field, all the above under the assumption of analyticity.

There can be analytical solutions existing for the BVP that are true solutions to the

two-plate problem, examples readily obtained, but they are of a limited topological variety. In cases of boundary conditions (69) and (70) for which no analytical solutions exist, we must turn to the so called weak mathematical solutions to the BVP, allowing for a discontinuous field solution embedding TDs in equilibrium with the embedding force-free field by having the magnetic pressure B^2 continuous across each TD. It is in this class of weak solutions that we find the TD bearing force-free fields of the two-plate problem, each case defined by a reference field \mathbf{B}_{neq} bearing the prescribed topology \mathcal{T} . We have thus shown an equivalence of the two-plate problem not with the BVP, but with the analytical BVP extended to allow for weak solutions.

5.5. The Parker infinitesimal phase-space of force-free fields around \mathbf{B}_0

To avoid the formidable nonlinearity of the force-free equations (2) and (3) and obtain the first mathematical proof of the Theorem, Parker (1972) considered the phase space of force-free fields in a small-parameter ϵ departure from \mathbf{B}_0 ,

$$\mathbf{B} = \mathbf{B}_0 + \sum_{n=1}^{\infty} \mathbf{B}_n \epsilon^n, \quad (74)$$

subject to boundary conditions $B_z = B_0$ at $z = 0, L$ that yield

$$z = 0, L, \quad B_{n,z} = 0, \quad \text{for all } n. \quad (75)$$

Taking \mathbf{B}_n to be solenoidal for all n , substituting power-series (74) into force-free equation (2) and then computing

$$\nabla \times [(\nabla \times \mathbf{B}) \times \mathbf{B}] = 0, \quad (76)$$

we obtain a power series in ϵ equated to zero. Setting the coefficients of all powers of ϵ to zero, we next obtain an infinite set of nonlinear PDEs relating \mathbf{B}_n of each order n to all \mathbf{B}_i or order $i < n$, subject to boundary conditions (75).

A natural meaning of the power-series parameter is $\epsilon = l/L \ll 1$ where l is a characteristic scale of the fpt displacement $\mathcal{D}_{z=0}$ on $z = 0$. Generally, for a sufficiently large L , the ϵ force-free field is expected to have an interior main part that connects to the boundaries $z = 0, L$ through two respective boundary layers of thicknesses of the order of l . If TDs must exist in the field under the Theorem, then the TDs must extend along flux surfaces from one boundary all through the interior to the other boundary. Let us assume the field (74) is analytic. Then, the nature of the field in $0 < z < L$ is instructive for parametrically large L .

Parker (1972) proved by mathematical induction that \mathbf{B}_n must be independent of z to all order n in the parametric limit of $L \rightarrow \infty$, as a necessary condition for force-free

equilibrium. This result is not physically acceptable because a fpt displacement $\mathcal{D}_{z=0}$ on $z = 0$ that continuously twists flux tubes out of uniform field \mathbf{B}_0 and braiding the twisted tubes about each other, will create a z dependence in the field not removable under the frozen-in condition. The inevitable conclusion then follows that analyticity and the power-series cannot be assumed, in general. The severity of the requirement of the independence of z for equilibrium to exist points to a propensity for a field of complex topologies to form TDs when allowed to dynamically relax to a force-free state under the frozen-in conditions. The Parker proof remains a unique insight into the inevitability of TDs in force-free fields treated in generality in the previous subsection.

5.6. An analytical force-free integral and the Parker Eikonal

The Parker (1992) optical analogy presents a tidy direct proof of the Theorem which we will later re-state with a connection to the (U, V) BVP.

The force-free equations (2) and (3) are expressed equivalently by equations (31) and (32) that introduce $\alpha(\mathbf{x})$ as the proportionality between the parallel current density and field. The $\alpha(\mathbf{x})$ -description casts in the (U, V) representation has an analytical integral found by Low (2005) that leads to the Parker Eikona ϕ by the treatment presented below.

By equation (32), the constant- α level surfaces are flux surfaces, so we may express $\alpha(\mathbf{x}) \equiv \alpha(U, V)$ varying in space as a strict function of the two flux functions. Rewrite PDE (31) as

$$\nabla \times \mathbf{B} = \alpha(U, V)\mathbf{B} \quad (77)$$

and define

$$\alpha(U, V) = \frac{\partial}{\partial U}\xi(U, V) - \frac{\partial}{\partial V}\zeta(U, V), \quad (78)$$

with no loss of generality. By the field representation $\mathbf{B} = \nabla U \times \nabla V$, equation (77) gives

$$\begin{aligned} \nabla \times [\nabla U \times \nabla V] &= \left[\frac{\partial \xi}{\partial U} - \frac{\partial \zeta}{\partial V} \right] [\nabla U \times \nabla V] \\ &= \nabla \times (\xi \nabla V + \zeta \nabla U), \end{aligned} \quad (79)$$

which has the general integral obtained by undoing the curl operation $\nabla \times$:

$$\nabla U \times \nabla V = \xi \nabla V + \zeta \nabla U + \nabla \psi, \quad (80)$$

where ψ is a free potential which we shall later relate to the Parker eikonal ϕ in PDE (56). It is important to note that this integration assumes an analytic field \mathbf{B} for which current density $\nabla \times \mathbf{B}$ contains no discrete CSs expressible as Dirac δ -functions.

Vector PDE (80) describes a force-free field in terms of the variables (U, V, ψ) for known (ξ, ζ) . Since (U, V) governed by PDEs (13) and (14) are sufficient for defining the force-free field, (ψ, ξ, ζ) are derived rather than defining properties of the force-free field. For each analytical (U, V) solution to the BVP, its attributes can be computed *a posteriori*, namely, the proportionality $[\xi(U, V), \zeta(U, V), \alpha(U, V)]$ and potential ψ .

The potential ψ is determined by the Poisson PDE

$$\nabla^2 \psi + \nabla \cdot (\xi \nabla V + \zeta \nabla U) = 0, \quad (81)$$

obtained by taking the divergence operation $\nabla \cdot$ across PDE (80), subject to boundary conditions consistent with $B_z = B_0$ at $z = 0, L$. The left hand side of PDE (80) is just the force-free field \mathbf{B} , so the boundary conditions on ψ are

$$z = 0, L, \quad \frac{\partial \psi}{\partial z} = B_0 - \left[\xi \frac{\partial V}{\partial z} - \zeta \frac{\partial U}{\partial z} \right]_{z=0, L}, \quad (82)$$

that has a unique solution for each analytical (U, V) solution of the BVP.

A clarification is in order before proceeding. As pointed out in Section 4.1.1, the representation $\mathbf{B} = \nabla U \times \nabla V$ involves a choice of the pair (U, V) with a freedom to express the same field \mathbf{B} in terms of another pair $[F(U, V), G(U, V)]$ as two independent functions of two variables. A change of flux-function description from (U, V) to (F, G) will self-consistently change the mathematical forms of PDEs (80) and (81) and boundary conditions (82). Physical properties are independent of the free choice of the flux-function pair in use, so those properties derived from PDE (80) must be rendered choice-independent to be sure they are of a physical nature.

With $\mathbf{B} = \nabla U \times \nabla V$, every prescribed function $F(U, V)$ set to a constant identifies a flux surface expressed by the identity:

$$\mathbf{B} \cdot \nabla F [U(\mathbf{x}), V(\mathbf{x})] = \mathbf{B} \cdot \left[\frac{\partial F}{\partial U} \nabla U + \frac{\partial F}{\partial V} \nabla V \right] \equiv 0, \quad (83)$$

assuring that the right hand side of PDE (80) has no component perpendicular to all prescribed constant- F flux surface. Therefore, on any such flux surface, $U(\mathbf{x})$ varies as a strict function of $V(\mathbf{x})$ and PDE (80) takes the form

$$\begin{aligned} \mathbf{B} &= \nabla U \times \nabla V \\ &= \left\{ \xi [U(V), V] + \zeta [U(V), V] \frac{dU(V)}{dV} \right\} \nabla V + \nabla \psi. \end{aligned} \quad (84)$$

Define the function $S(V)$ on such a constant- F flux surface by

$$\frac{dS(V)}{dV} = \xi [U(V), V] + \zeta [U(V), V] \frac{dU(V)}{dV}, \quad (85)$$

to rewrite equation (84) as

$$\mathbf{B} = \nabla [S(V) + \psi], \quad (86)$$

showing the force-free field is a potential field on the constant- F flux surface,

$$\begin{aligned} \mathbf{B} &= \nabla \phi, \\ \phi &= S(V) + \psi, \end{aligned} \quad (87)$$

where we recognize ϕ is the Parker Eikonal. Since ϕ is related directly to \mathbf{B} , we are assured that ϕ is physical, independent of the particular choice of the (U, V) representation in use. Corresponding to the fact that the right hand side of PDE (80) has no component perpendicular to all prescribed constant- F flux surface, the Parker Eikonal has no gradient perpendicular to any constant- F flux surface.

The Parker optical analogy follows from equation (87) giving the eikonal equation

$$|\nabla \phi|^2 = B^2, \quad (88)$$

describing the field lines on a constant- F flux surface as light rays in an optical medium of a refractive index $|\mathbf{B}|$. In a highly-twisted 3D force-free field, equilibrium is attained with spatially-localized regions of high magnetic pressure B^2 confined by the magnetic tension force. The force-free field lines are thus refracted *away* from such localized regions, high magnetic pressure corresponding to high refractive index. In the presence of sufficiently high magnetic pressures, refraction produces exclusions of the field lines from one or more of these regions, creating finite-sized holes or gaps in a stack of foliated flux surfaces, that do not exist when the field is not in equilibrium. Such a tunnel of gaps through the foliated stack may be punctured by flux systems external to and on either sides of the stack that have pushed inward, under the frozen-in condition, to make physical contact as a dynamical means of achieving a $\delta\mathcal{E} = 0$ force-free state. The contact surfaces are a complexity of flux surfaces, separating distinct flux systems, namely, the external flux systems pushing into the tunnel as well as the foliated fields in the stack. These contact surfaces naturally become TDs as a means of preserving the field topology in each flux system, subject to continuity of B^2 across each TD in equilibrium. The optical analogy of TD formation in 3D complex fields relaxing to equilibrium was discovered by Parker (1992, 1994). Here we add two more essential points to the optical analogy.

In all the non-equilibrium states continuously deformed from the uniform field \mathbf{B}_0 , the (U, V) flux surfaces have no gaps, each surface continuously filled by the deformed, anchored field lines of the original uniform field. In this case, the gaps through a stack of the deformed flux surfaces are punctured by external, highly-twisted, flux-tubes, every tube anchored to the boundaries, pushing their ways into one another from the two sides of a local stack of

twisted flux. Therefore, where gaps do form in the force-free field in the two-plate problem, the field is no longer continuous everywhere. Here it is important to recognize that the field-line exclusion from zones of high magnetic pressure is a localized, not a global effect.

An analytical (U, V) solution of the BVP has the propensity of containing flux systems detached from the boundaries $z = 0, L$. The presence of these flux systems, each closed within the domain $0 < z < L$, can exhibit gaps without the global field being discontinuous everywhere. But, of course, the closed flux systems by their presence mean that such a field is not a true solution of the two-plate problem.

The Theorem is well explained in terms of the two-plate problem being distinct from the (U, V) BVP, the latter an interesting mathematical problem in its own right but poses a different piece of physics. The two plate problem seeks a force-free field of a prescribed topology, with no interior closed flux system. The BVP seeks a field demanded by the point-wise imposition of the force-free condition in the domain, the mathematics of which allowing for a lax demand on field topology. An identical magnetic connectivity between the two plates is imposed in both problems, but the full topology of the field solution is free to adjust in the case of the BVP, in order to meet the force-free condition everywhere in the domain. In this case, everywhere force-free as a condition can be met, for some topologies or possibly for no topology at all.

Now, the BVP can be extended beyond its analytical solutions to admit the so-called weak solutions, allowing for a discontinuous \mathbf{B} which means discontinuity in the first partial derivatives of (U, V) such that TDs are admissible, subject to continuity of B^2 across each TD. Its extension to weak solutions renders the BVP equivalent to the two-plate problem. In this equivalence, we construct all the admissible solutions, the analytical and weak solutions that meet the same boundary conditions (69) and (70), found in the various topologies realized in them. The true solution for the equivalent two-plate problem must then be found among these solutions.

6. The restless coronal magnetic fields

The Theorem includes the far reaching statement that the force-free field of the two-plate problem must embed TDs *for most field topologies*. A follow-up interesting point is that, if resistivity is extremely weak but not zero, TDs must form under the perfect conductivity approximation, only to eventually and certainly dissipate by magnetic reconnection. The breakdown of the frozen-in condition endows the reconnected field with a different reconnected topology. Here lies the significance of the Theorem that *most* topologies are not

compatible with a continuous force-free state. That is, the reconnected field with the TDs removed does not merely return to the frozen-in condition. Its acquired topology is as likely to result in inevitable TDs and resistive reconnection as the the topology of the pre-connected field. The physical picture thus emerges of a complex 3D field ever in a state of TD formation and field reconnection, occurring intermittently in time and location.

A clarification is in order. We have treated spontaneous TD formation in a field of a fixed topology anchored to the rigid boundaries $z = 0, L$. The force-free state is the least complicated form of force-balance in the perfectly conducting fluid. The Parker Theorem is not strictly about static equilibrium. It says more, that the frozen-in condition is so far reaching that, striped of the complexity of time-dependent hydromagnetic processes, attaining an equilibrium state consistent with this fundamental property must involve the creation of TDs or CSs.

Let us continue with the previous discussion on the state of intermittent TD formation and dissipation. There is only a finite amount of magnetic energy in the anchored field. Each TD-driven field reconnection, via an extremely weak resistivity, changes the field's topology with a reduction of the total magnetic energy. The available amount of energy for such a resistive release is limited, of course. In the extreme case, when the field becomes potential, there is no free energy left. By the Theorem, the attainment of the potential state or any higher-energy meta-stable state is a endlessly tortuous path of TD formation and field reconnection. The limited amount of free energy for release, in the case of the anchored field, only means that TDs form successively with monotonically decreasing intensities (Janse & Janse 2004). In other words, while TD formation is inevitable, it is not imperative that the TDs must form with a lot of magnetic energy to dissipate.

The rigid boundary conditions of the two-plate problem studied can be replaced with a slow turbulent displacements of the magnetic foot-points that drive a quasi-steady evolution of the macroscopic field with an ubiquity of small-scale TD formation and dissipation. The level of energy release may be just enough to maintain the fluid at a million degree temperature, the ubiquitous nano-flares heating the corona (Parker 1988, 1994).

The coronal base is actually a free boundary across which fresh magnetic fluxes emerge at all scales from the Sun's interior hydromagnetic dynamo in cycles of 22 years. The emerged fluxes self-organize into relatively long-lived structures with nano-flaring to maintain the million-degree coronal environment. From time to time, such macroscopic fields may lose its meta-stability to reconfigure or be ejected out of the corona. In such high-energy events, large-scale intense TDs may form and dissipate via impulsive magnetic reconnection as major flares and Coronal Mass Ejections (Low 1996, 2002, 2018).

In both the nano and macro-scaled phenomena, TD formation is the only means of spontaneously creating resistive dissipation of magnetic field in spite of the corona's high conductivity. One may say, TD formation and dissipation by its physical nature is actually a high conductivity effect. The physically comprehensible phenomena in the corona are rooted in the Theorem as a fundamental property of the perfectly conducting fluid.

7. Erratum

Pages 1-17 constituting the original first version of this Conversation XII completed on November 23, 2020. Here the first version comprising Sections 1 - 4, pages 1- 17 is kept unchanged as a record. The added materials in the current version comprise Sections 5, 6, & 7, pages 18 - 35, the last being the Erratum here correcting 2 sets of incorrect equations.

7.1. PDEs for (U, V) - pages 3 & 4

On pages 3 and 4, the two pairs of equations (13) & (14) and (17) & (18), are incorrect, a source term is missing from each equation.

Sufficient to state that both pairs of equations should be replaced by the following correct equations

$$\nabla \cdot (B^2 \nabla U) = (\mathbf{B} \times \nabla V) \cdot \nabla (|\nabla U|^2) - (\mathbf{B} \times \nabla U) \cdot \nabla (\nabla U \cdot \nabla V), \quad (89)$$

$$\nabla \cdot (B^2 \nabla V) = (\mathbf{B} \times \nabla V) \cdot \nabla (\nabla U \cdot \nabla V) - (\mathbf{B} \times \nabla U) \cdot \nabla (|\nabla V|^2). \quad (90)$$

This corrected equivalent pair (89) & (90) is as originally intended only to view certain complicated terms in (U, V) in these equations directly in terms of \mathbf{B} and B^2 .

The governing equations for (U, V) are tidiest as given by PDEs (11) & (12) strictly in terms of (U, V) , to define the BVP, that we reproduce as (72) & (73) in Section 5.4 on page 27 for the convenience of reading Section 5.

7.2. The Parker eikonal equation, page 15

The Parker eikonal equation (56) has a missing source term inserted in the corrected version below:

$$[\nabla \phi(\xi_1, \xi_2)]^2 = \left(\frac{1}{h_1} \frac{\partial \phi}{\partial \xi_1} \right)^2 + \left(\frac{1}{h_2} \frac{\partial \phi}{\partial \xi_2} \right)^2 = B^2. \quad (91)$$

Spontaneous current-sheets in solar and stellar coronae

Crafoord Symposium Lecture

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(Draft April 15, 2021)

1. Introduction

The million-degree hot, fully ionized state of the X-ray radiating corona is sustained by ubiquitous *in situ* resistive dissipation of magnetic energy. This tenuous atmosphere conducts electricity and thermal heat efficiently and is dynamically structured by its global magnetic field of about 10 Gauss. Driven to preserve its natural tangled topology, 3D coronal fields tangentially form narrow sheets of electric current density, resulting in dissipation at the otherwise negligible weak resistivity. Thus, small-scale intermittent events of resistive heating are an essential component of the large-scale quasi-static corona. This irrepressible property of the observed corona has its origin in the magnetohydrodynamic (MHD) balance of forces in 3D fields^{1,2}.

This Lecture explains the physics of current sheet (CS) formation in strong magnetic fields as a high-conductivity process. This process implies that the corona is a universal stellar phenomenon, confirmed by the observations of the Chandra X-ray Observatory; launched into orbit in 1999, that the stars in our Galaxy commonly possess X-ray coronae.

2. The MHD corona

The fully ionized plasma as an MHD fluid is governed by the equations:

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p - \rho \nabla \Phi + \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B} + \nu_1 \nabla^2 \mathbf{v} + \left(\nu_2 + \frac{1}{3} \nu_1 \right) \nabla (\nabla \cdot \mathbf{v}), \quad (1)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}, \quad (2)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (3)$$

$$\frac{\partial s}{\partial t} + (\mathbf{v} \cdot \nabla) s = S, \quad (4)$$

where the dependent variables ρ , \mathbf{v} , p , \mathbf{B} are, respectively, density, velocity, pressure, magnetic field, Φ is the gravitational potential, the coefficients of viscosity (ν_1, ν_2) and electrical resistivity (η) are taken as constants. The set of equations is closed by the entropy s and its source S regarded formally to be functions of the dependent variables. In particular, the fluid temperature T as a function of p and ρ is accounted for by the source S .

Maxwell's equations are Lorentz invariant, the speed of light c being a universal constant independent of inertial frames. Whereas, Newtonian mechanics is non-relativistic, Galilean invariant between inertial frames. The MHD equations³ thus cannot carry the universal constant c , keeping only leading-order electromagnetic effects under the condition $\frac{|\mathbf{v}|}{c} \ll 1$. Accordingly, Lorentz transforming electric and magnetic fields (\mathbf{E} , \mathbf{B}) from the rest frame of a moving fluid element to the laboratory frame shows that both frames observe the same \mathbf{B} and current density \mathbf{J} related by Ampere's law,

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J}, \quad (5)$$

the Maxwell displacement current being negligible. Here c arises from using cgs units. Whereas, Ohm's law under Lorentz transformation gives the laboratory electric field

$$\begin{aligned} \mathbf{E} &= -\frac{1}{c} \mathbf{v} \times \mathbf{B} + \frac{1}{\sigma} \mathbf{J}, \\ &= -\frac{1}{c} (\mathbf{v} \times \mathbf{B} - \eta \nabla \times \mathbf{B}), \end{aligned} \quad (6)$$

where conductivity σ is related to the given resistivity by $\eta = \frac{c^2}{4\pi\sigma}$.

Provided magnetic gradients are moderate, the electric field \mathbf{E} is of order $\frac{|\mathbf{v}|}{c} |\mathbf{B}|$ implying a charge separation $\nabla \cdot \mathbf{E}$ of the same order. The electric force of order $\frac{|\mathbf{v}|^2}{c^2}$ is then negligible, so momentum eqn. (1) retains only the non-relativistic Lorentz force

$$\mathbf{L} = \frac{1}{c} \mathbf{J} \times \mathbf{B} = \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B}. \quad (7)$$

The leading order of $c|\mathbf{E}|$ is not negligible, so induction eqn. (2) follows from Faraday's law

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}. \quad (8)$$

In the MHD description, eqns. (1)-(4) are a complete set of equations for $(\rho, p, \mathbf{v}, \mathbf{B})$ that determine (\mathbf{E}, \mathbf{J}) as complementary variables given by eqns. (5) and (6).

2.1. Near-perfect electrical conductivity and current sheets

Consider the Spitzer electrical conductivity $\sigma = 2 \times 10^7 T^{3/2} s^{-1}$ in the corona treated as a fully ionized hydrogen gas. At $T = 2 \times 10^6 \text{ }^\circ K$, $\sigma = 6 \times 10^{16} s^{-1}$ and $\eta = 1.3 \times 10^3 \text{ cm}^2 s^{-1}$.

Although a poorer conductor than copper; $\sigma_{CU} = 6 \times 10^{17} \text{ s}^{-1}$ at room temperature, the corona is a near-perfect conductor by its astronomical size. A magnetic field of gradient scale L diffuses resistively at characteristic speed $u_\eta = \frac{\eta}{L} \text{ cm s}^{-1} < 1.3 \times 10^{-4} \text{ cm s}^{-1}$ with a timescale $t_\eta = \frac{L^2}{\eta} > 3 \times 10^3 \text{ years}$ for L larger than 10^2 km as an estimate of the corona's mean free path L_{mfp} . The magnetic Reynolds number $R_m = \frac{u_0}{u_\eta} = \frac{u_0 L}{\eta}$ is a dimensionless comparison of the induction and resistive terms on the right side of eqn. (6), u_0 being a characteristic fluid speed. For macroscopic phenomena, $L \gg L_{\text{mfp}}$ and taking u_0 to be a sound speed 10^2 km s^{-1} , the magnitude $R_m \geq 10^{10}$ shows that the corona as a fluid is effectively a perfect conductor, resistive diffusion quite negligible.

This linear reasoning overlooks the nonlinear coupling between momentum and induction eqns. (1) & (2). Express the Lorentz force as

$$\frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B} = -\nabla \left(\frac{1}{8\pi} B^2 \right) + \frac{1}{4\pi} (\mathbf{B} \cdot \nabla) \mathbf{B}, \quad (9)$$

the sum of an isotropic-pressure force and a tension force. The tenuous corona is dominated by its magnetic field of about 10 Gauss, characterized by the dimensionless pressure ratio $\beta = \frac{8\pi p}{B^2} \ll 1$. Under the joint conditions $\beta \ll 1$ & $R_m \gg 1$, a 3D field readily drives flows that locally steepen field gradients, expelling magnetic flux out of the way for two fields of distinct orientations to come into tangential contact. A tangential discontinuity (TD) in \mathbf{B} , containing a CS, is then created. This localized, single-event, nonlinear process is distinct from the well-known global cascade of velocity scales down to viscosity-important small scales in incompressible hydrodynamic turbulence.

In the formation of a TD, perfect conductivity with $\eta = 0$ should be distinguished from near-perfect conductivity with $R_m \gg 1, \eta \neq 0$. If $\eta = 0$, field-gradient steepening proceeds to tangential discontinuities (TDs). If $\eta \neq 0$, the steepening under $R_m \gg 1$ proceeds only to have the strong gradient removed resistively wherever $R_m \rightarrow 1$ for a sufficiently small gradient scale L . The ensuing current dissipation produces local heating and MHD turbulence as the fluid returns to $R_m \gg 1$. Individual local events cannot remove the field's three-dimensionality and they naturally recur under persisting conditions $\beta \ll 1, R_m \gg 1$, a basic property discovered theoretically with the two-plate problem¹.

2.2. The two-plate problem and frozen-in condition

Consider the following Cartesian initial-boundary value problem for a field in an unbounded fluid in $0 < z < L_0$, all field lines connecting two parallel, rigid, perfectly-conducting plate-boundaries at $z = 0, L_0$. As sketched in Fig. 1, this two-plate model geometrically

mimics an inverted-U tube of tangled, braided magnetic field in the corona with the tube's footprints rooted to the coronal base. For simplicity we neglect gravity and take the "coronal" fluid between the plates to be cold and perfectly-conducting, a case of $\beta \rightarrow 0$, $R_m \rightarrow \infty$. Eqn. (4) is irrelevant whereas the dependent variable $\mathbf{v}, \mathbf{B}, \rho$ are governed by eqns. (1)-(3) with $p = 0, \Phi = 0, \eta = 0$. Let us pose the problem associated with a uniform field $\mathbf{B}_0 = B_0 \hat{\mathbf{z}}$:

$$z = 0, L_0, \quad \mathbf{v} = 0; \quad B_z = B_0, \quad (10)$$

$$(x^2 + y^2)^{1/2} \rightarrow \infty, \quad \mathbf{v} \rightarrow 0; \quad \mathbf{B} \rightarrow \mathbf{B}_0, \quad (11)$$

$$t = t_1, \quad \mathbf{v} = 0; \quad \mathbf{B} = \mathbf{B}_1(\mathbf{x}); \quad \rho = \rho_1(\mathbf{x}). \quad (12)$$

For $B_0 \neq 0$, momentum and induction eqns. (1) & (2) independently demand for condition $\mathbf{v} = 0$ at $z = 0, L_0$. The initial static state $[\mathbf{B}_1(\mathbf{x}), \rho_1(\mathbf{x})]$ is prescribed subject to \mathbf{B}_1 having only field lines connecting $z = 0, L_0$. For the present, we assume mathematical analyticity.

With $\eta = 0$, induction eqn. (2) implies the frozen-in condition

$$\frac{d}{dt} F(\Sigma) \equiv \frac{d}{dt} \int_{\Sigma} \mathbf{B} \cdot d\Sigma = 0, \quad (13)$$

stipulating the conservation of the magnetic flux F across each fluid surface Σ deforming in a continuous flow \mathbf{v} . Therefore, the $F(\Sigma) = 0$ fluid surfaces coinciding with the flux surfaces of the initial field $\mathbf{B}_1(\mathbf{x})$ remain as flux surfaces of the evolving field $\mathbf{B}(\mathbf{x}, t)$. That is, the flux surfaces defined by $\mathbf{B}_1(\mathbf{x})$ permanently partition the fluid into a continuum of invariant sub-volumes, each containing a fixed mass of fluid within flux surfaces "frozen" into the fluid.

Each flux surfaces is continuously covered by non-intersecting, solenoidal field lines:

$$\frac{d\mathbf{x}}{dl} = \frac{\mathbf{B}}{|\mathbf{B}|}, \quad (14)$$

l denoting the field-line length. The field-line foot points are fixed at $z = 0, L_0$ where \mathbf{v} vanishes, defining a permanent one-to-one map $\mathcal{FP} : (x, y, 0) \leftrightarrow (x, y, L_0)$ between the plates. The partition sub-volumes, hereafter the B-tubes, generally are intertwined and braided tubes of magnetic flux extending from plate to plate, each tube with fixed boundary footprints. Driven by the Lorentz force, the complexity of anchored, tangled B-tubes push and yield with no exchange of fluid across the flux surfaces.

In the ensuing MHD evolution, the sum of kinetic and magnetic energies decreases monotonically, by the viscous dissipation of kinetic energy generated by the Lorentz force at the expense of magnetic energy, and, by the escape of MHD waves to the far reaches $(x^2 + y^2)^{1/2} \rightarrow \infty$. Under boundary condition (11), there is a finite amount of free energy in

the initial field $\mathbf{B}_1(x, y, z)$, measured above the uniform energy-density of \mathbf{B}_0 , to be drained by viscosity. The $t \rightarrow \infty$ outcome of the evolution is then a static force-free field:

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = 0, \tag{15}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{16}$$

meeting conditions (10) & (11), except when analyticity breaks down by $\mathbf{B}(\mathbf{x}, t)$ becoming discontinuous. This exception is, in fact, the rule for 3D fields for which an analytical force-free field is generally unavailable as an end-state to the MHD evolution².

2.3. Current-sheets and the Optical Analogy

TDs result from a 3D geometric failure of the B-pressure and B-tension forces to align at each point in space; see eqn. (9). The former acts normal to a constant-pressure surface whereas the latter lies in the curvature plane of the field line through the point, the alignment generally not possible everywhere in a continuous 3D field globally frozen into the fluid.

By its textbook definition the topology of a given set of geometric objects comprises all those properties that are meaningful and invariant under *all continuous* deformations. In context here are the $F(\Sigma) = 0$ fluid surfaces as geometric objects deforming in a continuous velocity \mathbf{v} driven by the Lorentz force. The $F(\Sigma) = 0$ fluid surfaces by their mutual intersections define the plate-anchored field lines, and, as partition boundaries define the B-tubes, all anchored to the plates with a topologically invariant foot-point map \mathcal{FP} . Denote by $\mathcal{M}(\tau)$ the manifold of *all* continuous fields satisfying boundary conditions (10) & (11) that are topologically equivalent, the notation τ indicating these fields are continuously deformable one into another in the manifold. Thus, the initial field $\mathbf{B}_1(x, y, z)$ defines the manifold $\mathcal{M}(\tau)$ wherein the evolution of $\mathbf{B}(\mathbf{x}, t)$ traces a path of decreasing sum of kinetic and magnetic energies. Any other initial field $\mathbf{B}'_1(x, y, z)$ topologically equivalent to $\mathbf{B}_1(x, y, z)$ belongs to $\mathcal{M}(\tau)$ where it initiates a generally different evolutionary path. Whereas, if $\mathbf{B}_1(x, y, z)$ and $\mathbf{B}'_1(x, y, z)$ are not topologically equivalent, $\mathbf{B}'_1(x, y, z)$ belongs to a different manifold $\mathcal{M}(\tau')$ expressed by $\tau' \neq \tau$. The initial density $\rho_1(x, y, z)$ has no role in this topological view of the evolution, other than defining the fixed amount of mass in each B-tube and the fluid inertia by which the Lorentz force drives velocity \mathbf{v} .

The Lorentz force can readily force two B-tubes into a flux-surface contact. The two B-tubes may not be initially contiguous but have made contact within an opening or a gap forced through a layer of flux between them. A complexity of contact flux-surfaces then develops dynamically, keeping all the tangled B-tubes distinct as they push and yield, in response to the viscous loss of energy and under the frozen-in condition. Any contact surface

separating two B-tubes may become a TD where two fields from different flux systems meet tangentially. Subject to the force-balance requirement of a continuous B-pressure $\frac{1}{8\pi}B^2$ across the TD, the discontinuous fields have the same magnitude but unequal orientations to satisfy the frozen-in condition in their respective flux systems. Developing a TD proceeds in a manifold $\mathcal{M}(\tau)$ until analyticity fails by $\mathbf{B}(\mathbf{x}, t)$ becoming discontinuous. Further viscosity-driven evolution in time is still physically meaningful with $\eta = 0$, requiring extending the governing MHD equations to dynamically account for TDs as discrete structures in the field. The frozen-in condition remains but the invariant B-tubes now have flux-surface boundaries that may contain TDs.

The Optical Analogy⁴ describes TDs in 3D static force-free fields by an application of Fermat's Principle governing light rays in a refractive medium. Assuming analyticity, the force-free field and its field-aligned current density lie on common flux surfaces. Take a flux surface Σ with orthogonal coordinates (ξ_1, ξ_2) and distance metric $dl^2 = h_1^2 d\xi_1^2 + h_2^2 d\xi_2^2$, $h_1(\xi_1, \xi_2)$ and $h_2(\xi_1, \xi_2)$ being the Lamé coefficients. The current density $\nabla \times \mathbf{B}$ is tangential to Σ , so the force-free field \mathbf{B} is potential in Σ :

$$\mathbf{B} = [B_{\xi_1}, B_{\xi_2}] = - \left[\frac{1}{h_1} \frac{\partial \phi_\Sigma}{\partial \xi_1}, \frac{1}{h_2} \frac{\partial \phi_\Sigma}{\partial \xi_2} \right]. \quad (17)$$

Given the force-free field, its potential ϕ_Σ is determined by an eikonal equation with B as refractive index, and its field lines in Σ are determined by eqn. (14) in terms of the potential:

$$\frac{1}{h_1^2} \left(\frac{\partial \phi_\Sigma}{\partial \xi_1} \right)^2 + \frac{1}{h_2^2} \left(\frac{\partial \phi_\Sigma}{\partial \xi_2} \right)^2 = B^2 \quad (18)$$

$$\begin{aligned} & , \quad B_{\xi_1} h_1 \frac{d\xi_1}{dl} = - \frac{1}{h_1} \frac{\partial \phi_\Sigma}{\partial \xi_1}, \\ & \quad B_{\xi_2} h_2 \frac{d\xi_2}{dl} = - \frac{1}{h_2} \frac{\partial \phi_\Sigma}{\partial \xi_2}. \end{aligned} \quad (19)$$

Eikonal equation (18) is nonlinear, B and ϕ_Σ related by force-free eqns. (15) & (16). The analogy of the force-free field lines as refracting light rays in Σ has an instructive implication.

Streaming field lines in Σ refracting around a localized B maximum are naturally concave inward for the B-tension force to balance the outward B-pressure force. A sufficiently steep B-pressure gradient may completely exclude the field lines on Σ from a finite-size region around the maximum that is actually a hole in the surface. The holes of a foliated stack of Σ -like flux surfaces thus constitute a 3D tunnel through which fluxes external to the stack intrude at both ends of the tunnel to make contact. The essential point here is that the tunnels are ready features of the continuous parts of a 3D force-free field, by the Optical Analogy. Whereas, the pressing of the B-tubes into one another naturally punctures holes in

flux surfaces in the course of monotonic viscous loss of energy. A 3D field starts evolving in a manifold $\mathcal{M}(\tau)$ with all its plate-to-plate flux-surfaces *everywhere covered* by plate-to-plate field lines but generally would end up with tunnels punched through layers of flux surfaces by discontinuous fluid displacements favored by the energy-decreasing evolution.

This inevitability of TDs has its origin in a fundamental magnetostatic theorem¹ developed in my monograph². Mathematical analysis shows that a 3D $\mathcal{M}(\tau)$ is generally not a compact manifold⁵, like the open unit interval $0 < x < 1$ that does not include the end points. If $\mathcal{M}(\tau)$ contains a minimum-energy force-free field among its topologically-equivalent, 3D continuous fields, this minimum-energy state is, of course, the end-state of an energy-minimizing evolution with no loss of analyticity. If $\mathcal{M}(\tau)$ contains no force-free field, the general case, the minimum-energy state still exists by the evolving field running out of energy. The end-state is then a TD-embedding force-free field that lies not in $\mathcal{M}(\tau)$ but at the "open" boundary of the manifold. The viscosity-driven evolution approaches this discontinuous end-state from within $\mathcal{M}(\tau)$. This is a mathematically beautiful problem in variational calculus.

2.4. Evolution under near-perfect conductivity

In the corona where $\eta \neq 0$ is small, TDs develop under $R_m \gg 1$ to unavoidably dissipate by the weak resistivity. Whereupon, the frozen-in preservation of the B-tubes breaks down, with significant global effects. Locally the field lines reconnect resistively as steep field gradients are removed. The foot-point map \mathcal{FP} , no longer preserved, is thus redefined by the reconnected field lines in a return to the $R_m \gg 1$ condition. The flux-surfaces topology has changed but is re-frozen in the fluid, defining a new magnetic partition of the fluid into B-tubes. The simplicity of evolution in a fixed manifold $\mathcal{M}(\tau)$ is lost, but the basic irrepressible 3D propensity for TD formation under $R_m \gg 1$ remains, driving a more complex $R_m \gg 1$ evolution continually interrupted by episodes of TD formation and dissipation throughout the fluid. This is the basic MHD nature of coronal heating.

The initial-boundary value problem posed by rigid conditions (10)-(12) serves to demonstrate spontaneous CSs with a minimum of physical complication. The magnetic field in the corona is not rigidly anchored, extending up from the $5,000^\circ K$, heavy, convective solar surface called the photosphere where $\beta \approx 1$. The two atmospheric layers are separated by the chromosphere, thin at only a few percent of the solar radius $R_\odot = 7 \times 10^5 \text{ km}$, up through which temperatures rise by 2 orders to million degrees as densities fall by 8 orders. Photospheric convective granules, typical size of 300 km and speed of 0.5 km s^{-1} , radiate away the Sun's white-light luminosity $L_\odot = 3.9 \times 10^{33} \text{ erg s}^{-1}$ clear through the tenuous corona.

The photosphere turbulently tangles the coronal field lines under conditions of $R_m \gg 1$, slowly stressing the coronal fields to adjust dynamically in near-equilibrium at large $\beta \ll 1$ MHD wave speeds of about 10^3 km s^{-1} .

The spontaneous CSs and their dissipation are sustained dynamically by the Poynting flux of magnetic energy pumped upward by the photosphere. The observed corona is perennially populated by a broad spectrum of small-scale, ubiquitous heating events, notably the nano-flares each liberating an estimated $3 \times 10^{23} \text{ erg}$. Identifying these events with resistive magnetic reconnection and heating due to spontaneous CSs, a heat input of the order of $10^5 - 10^7 \text{ erg cm}^{-1} \text{ s}^{-1}$ through the corona base may be estimated, sufficient to explain the observed $10^{27} - 10^{28} \text{ erg s}^{-1}$ low-energy X-ray radiated thermally by the corona, and account for an order of magnitude larger thermal energy loss by thermal conduction downward to be efficiently radiated by the chromosphere and outward to the upper atmosphere expanding into the solar wind.

Photospheric convective motions working on the coronal field is the source of heat input which is released ubiquitously and intermittently by irrepressible spontaneous CSs in the global corona.

3. Concluding remark

The coronal poses a varied range of plasma physics problems, each challenging in its own right; plasma and MHD stabilities & turbulence, macroscopic self-organized magnetic structures & their eruptions, solar & stellar wind expansion, to mention a few. Spontaneous CSs is a conceptual discovery and understanding of the central physical question posed by the universal phenomenon of solar/stellar coronae, how a highly electrically conducting atmosphere is able to efficiently heat itself resistively.

Acknowledgment

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The Corona and Solar Wind

Crafoord Public Lecture

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The Sun, as one of billions of magnetically active stars in our Galaxy, is a natural laboratory to observe astrophysical plasma physics and magnetohydrodynamics (MHD). The million-degree hot, fully-ionized solar corona and its supersonic solar wind are a case in point. This Lecture describes in broad-brush physics why the Sun has a corona and why the corona is compelled to expand into the solar wind that fills all interplanetary space.

1. Introduction

The solar magnetic field, hereafter the field, is about 10 Gauss at the $5,000^\circ K$ solar surface, the photosphere, that thermally radiates almost steadily at the rate of $L_\odot = 3.9 \times 10^{33} \text{ erg } s^{-1}$. Sustained by dynamo actions in the solar interior, the field is actively variable on all observable timescales up to the well known eleven-year sunspot cycles. The global field reverses its North-South magnetic polarity within 3-4 years in each cycle. Dynamically dominated at the photosphere by turbulent fluid convection at speeds of about $0.5 \text{ km } s^{-1}$, the solenoidal field penetrates unabated into the $2 \times 10^6 \text{ }^\circ K$ corona, passing through the chromosphere of a thickness only a few percent of the solar radius $R_\odot = 7 \times 10^5 \text{ km}$. Temperature increases steeply through the chromosphere by 2 orders of magnitude as density decreases hydrostatically by 8 orders.

The mainly-hydrogen corona is tenuous and optically thin, dominated dynamically by its magnetic field and uninfluenced by the white-light luminosity $L_\odot = 3.9 \times 10^{33} \text{ erg/s}$ passing clear through it except for a negligible scatter that produces the coronal halo around the Sun in a total eclipse. The degree of ionization increases steeply through the chromosphere to complete ionization in the million-degree corona composed of free moving electrons, protons and ionized helium and other minority atomic species. Without the line emissions of neutral atoms, the corona at its million-degree temperature radiates thermally at about $10^{27} - 10^{28} \text{ erg } s^{-1}$ in extreme-ultraviolet and low-energy X-rays. Nowadays solar instruments in space above the terrestrial atmosphere allow us to directly image the corona in its short-wavelength radiations against the un-occulted cool photosphere; Fig. 1. Driven by the relatively slow moving heavy photosphere, the coronal fields restlessly adjust at large MHD

wave-speeds of the order of $10^3 km/s$ to remain close to quasi-static conditions except when interrupted by large-scale solar flares and episodic mass ejections.

The stage is thus set for my discussion of the heating of the corona and the mechanics of the solar wind. The generality of the physics presented shows that the corona and its solar wind must be a universal stellar phenomenon^{1,2,3,4}. This universality was confirmed by the discovery of the orbiting Chandra X-ray Observatory (launched 1999) that the stars in our Galaxy commonly possess X-ray coronae.

2. The highly conductive corona and its gravitational confinement

The chromospheric rise of temperature is a natural feature of hydrostatic atmospheres balancing heating against radiative loss via thermal conduction. Stable gravitational stratification favors a density decreasing with atmospheric height. Radiative loss by particle collisions has a rate quadratic in the density. An ubiquitous heating varying with density less strongly than quadratic would then produce excessive heating in the upper rarefied atmosphere. Naturally, temperature must increase monotonically with height to conduct excess heat downward to be radiated away by the dense lower atmosphere.

Empirical modeling of the complex interplay of atomic physics and radiative transfer under quasi-static conditions estimates a required heat input into the corona of the order of $5 \times 10^{29} \text{ erg s}^{-1}$, averaging $10^5 - 10^7 \text{ erg cm}^{-2} \text{ s}^{-1}$ over the coronal base. This total heat input is small compared to the white-light luminosity $L_{\odot} = 3.9 \times 10^{33} \text{ erg s}^{-1}$ but is 1 to 2 orders of magnitude larger than the $10^{27} - 10^{28} \text{ erg s}^{-1}$ coronal X-ray radiation to account for thermal conduction of heat too to be addressed shortly.

Free electrons render the fully ionized corona an almost-perfect electrical conductor, with a conductivity $\sigma = 2 \times 10^7 T^{3/2} \text{ s}^{-1}$ or, alternatively, a resistive diffusion coefficient $\eta = \frac{c^2}{4\pi\sigma} \text{ cm}^2 \text{ s}^{-1}$, in cgs units where c is the speed of light. At characteristic temperature $T = 2 \times 10^6 \text{ }^\circ \text{K}$, $\sigma = 6 \times 10^{16} \text{ s}^{-1}$ and $\eta = 1.3 \times 10^3 \text{ cm}^2 \text{ s}^{-1}$. Over a macroscopic length scale L , the field resistively diffuses with timescale $t_{\eta} = \frac{L^2}{\eta}$ to be compared to timescale $t_{u_0} = \frac{L}{u_0}$ associated with a characteristic macroscopic speed u_0 . The dimensionless magnetic Reynolds number $R_m = \frac{t_{\eta}}{t_{u_0}} = \frac{u_0 L}{\eta}$ is greater than 10^8 for most observable coronal phenomena. In this $R_m \gg 1$ high-conductivity environment, the field stores energy by its persisting electric current and behaves as though frozen into the fluid. A consequence of this frozen-in coupling between fluid and field is a spontaneous dynamical formation of thin sheets of current separating magnetic fluxes, fundamentally irrepressible in 3D complex fields⁴. These sheets form as thin as the conductivity is high, meaning that each sheet must dissipate

resistively as the local R_m falls to order unity, identifying L with the sheet thickness. Thus, a highly-conducting magnetized fluid not only stores magnetic energy but is also resistively self-heating by the dissipation of spontaneous currents sheets.

Spontaneous current-sheets explain the small scale nanoflares ubiquitous in the corona, each event liberating an estimated 3×10^{23} *erg* over about 10^2 seconds. A new nanoflare every 3 seconds can account for the required heat input into the corona of 10^7 *erg cm⁻² s⁻¹* averaged over its base^{4,5}.

Temperature rises with height to a maximum in the low corona from where it decreases with increasing heliocentric distance. Both conductive thermal fluxes are significant, downward to the efficiently radiating chromosphere and outward from the Sun. Here we encounter the other important property of the fully-ionized corona, namely, the large thermometric conductivity $\kappa = 3 \times 10^9 T^{5/2} N_e^{-1} \text{ cm}^2 \text{ s}^{-1}$ along the field due to the free electrons of number density N_e . At characteristic $T = 2 \times 10^6$ ° *K*, $N_e = 10^8 \text{ cm}^{-3}$ at mid corona, $\kappa = 1.7 \times 10^{17} \text{ cm}^2 \text{ s}^{-1}$. The conduction electrons along the field gyrate tightly around the field lines, the tightness reducing conduction across the field by a factor of about 10^8 . To clarify, heat is conducted by temperature diffusing with coefficient κ whereas electric current dissipates by the field resistively diffusing with coefficient η .

Let us ignore the presence of the field and its anisotropic influence on thermal conduction to make a simple physical point. High thermal conductivity at million-degree temperatures supplies heat far out by very many solar radii⁶. Solar gravity falls as the inverse-square of heliocentric distance r . The heat conducted to the outer layers of the static atmosphere can readily exceed the gravitational binding energies of these layers. In other words, these layers can break gravitational confinement and expand outward into the unbounded open space. The bulk speed at which each layer escapes gravity naturally becomes supersonic because of the rapid decrease of temperature and sound speed in the expansion flow. Thus, fluid actions in each supersonic layer can communicate at the local sound speeds only to its the outer layers and never back to influence the subsonic and almost-static inner atmosphere. The loss of mass by the supersonic part of the outflow is balanced by a subsonic upward flow of mass from the dense low corona. This is the basic mechanics of the supersonic solar wind, a consequence of heating and high coronal thermal conductivity.

My theoretical discovery of the solar wind¹ in 1957 was made in a comprehensive examination of observational and theoretical issues known at the time. The aurorae and other geomagnetic phenomena in the high latitudes of the Earth had been investigated extensively, highly suggestive of high-energy electrons and particles from the Sun. Cosmic rays of solar and galactic origins were being detected from high-altitude balloons, discovering temporal flux variations that correlated with the Sun's rotation and coronal phenomena. There were

also the observations of comets rounding the Sun with their long tails characteristically pointing away from the Sun along their orbits. It was estimated that the photon pressure of the Sun’s white-light is insufficient to account for the outward solar alignments of cometary tails⁷. Observations had estimated that the gas pressures in interstellar space beyond Pluto orbit is a factor of some 10^8 smaller than the coronal pressure.

These observations considered against the hydrodynamic nature of a heated, thermally-highly conducting, spherically-symmetric stellar atmosphere led me to the inescapable conclusion that the solar corona could not be everywhere gravitationally bound, but must globally expand outward, attaining supersonic speeds within a few solar radii in heliocentric distance and flowing past the Earth with an estimated speed of about 300 km s^{-1} . My discovery was timely at the threshold of the Space Era that began in 1957 with the first human-made satellite Sputnik in space. The existence of the solar wind was soon confirmed by *in situ* observations^{8,9} from Soviet satellite Luna 1 in 1959 and US satellite Mariner 2 in 1962.

Space plasma physics and MHD blossomed with great advancements in the six decades following, using generations of spacecrafts. Knowing theoretically why there should be the solar wind was a solid scientific footing for understanding its first space observations and the wealth of space discoveries that followed. We have observed the global solar wind evolving with the Sun in successive sunspot cycles and studied the solar-wind influence and interactions with the atmospheres of the planets and their moons, comets and asteroids. Fundamental issues on the origin of solar and galactic cosmic rays were resolved. The magnetic interaction between the Sun and solar wind over astronomical timescales at once offered an explanation for loss of angular momentum during star formation. Stars and galaxies also have expansive wind flows, now common astrophysical concepts.

Three events are noteworthy. Launched in 1977, the twin NASA spacecraft Voyagers I and II a few years ago passed through the solar-wind terminal shock, located beyond Pluto orbit, to subsequently enter interstellar space. The NASA Parker Solar Probe launched in 2018 travelled in the opposite direction to orbit the Sun within $10 R_{\odot}$ to observe the *in situ* coronal conditions where the solar wind originates. And, the NASA/ESA Ulysses spacecraft launched in 1990 into a polar orbit around the Sun had observed the solar wind at all solar latitudes.

3. Polytropic static atmospheres and expansive winds

The polytropic gas provides an elementary illustration of the global supersonic expansion as the natural state of a heated atmosphere.

Let us treat the fully-ionized hydrogen, comprising equal numbers of protons and electrons, as a fluid with density ρ defined by the much heavier protons and pressure p as the sum of equal partial pressures contributed by the protons and electrons. The reversible adiabatic compression or expansion of the fluid is governed by the equation of state

$$p = s_0 \rho^\gamma, \tag{1}$$

where adiabatic index $\gamma = 5/3$ and it is assumed that s_0 as a measure of the fluid's entropy is constant in space and time. This adiabatic property follows from the assumption that each fluid parcel is thermally isolated and suffers no heat addition or loss. The polytropic fluid is a mathematical model that adopts equation of state (1) but allowing γ to be a freely-prescribed constant parameter. The fully ionized hydrogen with $\gamma = 5/3$ thus serves as a physical reference case among the polytropes.

In response to an expansion, the pressure of a polytrope with index $\gamma > 5/3$ decreases with its density more strongly than in the reference case of $\gamma = 5/3$. Thus, under expansion, a $\gamma > 5/3$ polytrope qualitatively mimics the non-adiabatic expansion of the $\gamma = 5/3$ fluid associated with a heat loss such as by radiation. By similar consideration, an expanding $\gamma < 5/3$ polytrope mimics the non-adiabatic expansion of the $\gamma = 5/3$ fluid associated with a heat gain such as by thermal conduction. The $\gamma \neq 5/3$ polytropes are also related to the $\gamma = 5/3$ fluid by a corresponding set of comparative compressive behaviors.

We use the polytropic model to construct below an intuitive understanding of how successfully solar gravity may confine an atmosphere in static equilibrium, failing which a solar-wind expansion is the preferred steady state. For convenience we identify the polytropes by $\gamma = 1 + \frac{1}{n}$, $n > 0$, with $n = 3/2$ corresponding to the $\gamma = 5/3$ reference case.

3.1. Static atmospheres

The static equilibrium equation of a spherically-symmetric solar atmosphere,

$$\frac{dp}{dr} + \rho \frac{GM_\odot}{r^2} = 0, \tag{2}$$

subject to equation (1) with $\gamma = 1 + \frac{1}{n}$, has the general solution for the density:

$$\rho = \left[\frac{GM_\odot}{s_0(n+1)} \right]^n \left[\frac{1}{r} + E_0 \right]^n, \tag{3}$$

where r is the spherical radial coordinate, G the Newtonian gravitational constant, M_\odot the solar mass, and E_0 a constant parameter generating a family of solutions for a given n .

Consider the instructive theoretical problem seeking the static equilibrium of a given mass M_0 of fluid located in the unbounded space $r > R_\odot$, artificially treating the Sun as a rigid solid sphere. In the absence of gravity, the fluid would by its pressure expand out to $r \rightarrow \infty$ as its density tends to zero everywhere. In the case of a gravitating sphere of radius R_\odot , let us assume that the mass M_0 in its exterior is fully confined by gravity as described by solution (3), meaning no part of this mass has escaped to $r \rightarrow \infty$. The boundary condition $\rho \rightarrow 0$ at $r \rightarrow \infty$ then demands that $E_0 \leq 0$. We thus express the solution as

$$\begin{aligned} \rho &= \left[\frac{GM_\odot}{s_0(n+1)} \right]^n \left[\frac{1}{r} - \frac{1}{R_1} \right]^n; & R_\odot < r < R_1, \\ &= 0 & R_1 < r < \infty, \end{aligned} \quad (4)$$

redefining the parameter as $E_0 = -\frac{1}{R_1}$. The fully-confined atmosphere lies parametrically in the range $R_\odot < R_1 < \infty$. Each solution describes an atmosphere abutting vacuum at an atmospheric top $r = R_1$ where density and pressure vanish.

The parameter R_1 is determined by the transcendental equation

$$M_0 = 4\pi \left[\frac{GM_\odot}{s_0(n+1)} \right]^n \int_{R_\odot}^{R_1} \left[\frac{1}{r} - \frac{1}{R_1} \right]^n r^2 dr, \quad (5)$$

in terms of the prescribed total mass $M_0 > 0$. The interesting theorem then follows that there is no upper bound to M_0 if $n \leq 3, \gamma > 4/3$, including the reference case of $\gamma = 5/3$. In particular, $M_0 \rightarrow \infty$ as $R_1 \rightarrow \infty$ if $\gamma > 4/3$, showing that a fluid of any amount M_0 can be wholly confined gravitationally in the unbounded space $R_\odot > r$.

In contrast, M_0 has a finite upper bound

$$M_{max} = 4\pi \left[\frac{GM_\odot}{s_0(n+1)} \right]^n \frac{1}{n-3} \frac{1}{R_\odot^{n-3}} < \infty, \quad (6)$$

if $n > 3, \gamma < 4/3$. Since one may physically insert any amount of fluid M_0 into the unbounded space $R_\odot > r$, it follows that no static equilibrium state is available to the inserted fluid if $M_0 > M_{max}$, except by an ejection of mass out to infinity to leave behind a residual mass $\Delta M_0 \leq M_{max}$ in an equilibrium state described by solution (4). The gravitational escape speed falls as $r^{-1/2}$. A simple analysis, not necessary to repeat here, shows that the relatively slow $\gamma < 4/3$ polytropic decrease of p with density; implying heat gain relative to $\gamma = 5/3$, has sufficient energy to break gravitational confinement in the far radial region if $M_0 > M_{max}$. An upper layer of the atmosphere must then peel off and expand against gravity to infinity in an episodic ejection flow.

3.2. Steadily expanding winds

Let us leave the rigid-sphere model to treat a polytropic corona expanding steadily into interplanetary space at velocity v with a constant mass flux F_0 , described by the equations

$$\rho v \frac{dv}{dr} = -\frac{dp}{dr} - \rho \frac{GM_\odot}{r^2}, \quad (7)$$

$$r^2 \rho v = F_0. \quad (8)$$

By polytropic equation (1), momentum equation (7) has the integral

$$\frac{1}{2}v^2 + s_0(n+1)\rho^{\frac{1}{n}} - \frac{GM_\odot}{r} = E_0, \quad (9)$$

as a generalization of the static solution (3), E_0 a free constant. Given (F_0, E_0) , algebraic equations (8) and (9) can be solved simultaneously for a $[v(r), \rho(r)]$ wind-solution.

Sufficient for our purpose, we point out an essential property of these steady solutions describing a global polytropic flow in the domain $0 < r < \infty$ from the origin representing the Sun as a point source, discharging a given mass flux F_0 through a near-static subsonic atmosphere into a supersonic far flow with a terminal speed $v_\infty = \sqrt{2E_0}$. A necessary condition $\gamma \leq 3/2$ applies for such a wind solution to exist, arising from the energetics of the wind approaching a constant terminal speed. For $r \rightarrow \infty$, mass conservation requires $\rho \propto r^{-2}$ with a polytropic pressure $p \propto r^{-2\gamma}$ that must keep up with the gravitational potential-energy density $\rho \frac{GM_\odot}{r} \propto r^{-3}$, thus requiring $\gamma \leq 3/2$.

The polytropic sound speed given by $c_s^2 = s_0 \gamma \rho^{\gamma-1}$ monotonically decreases along the expansive flow as the flow speed accelerates. Thus, the flow naturally becomes supersonic at a point $r = r_s$ with no fluid influence passing from the supersonic $r > r_s$ region back to the subsonic $r < r_s$ region. Differential equation (7) may be rewritten as

$$(v^2 - c_s^2) \frac{dv}{dr} + \frac{v}{r} \left(\frac{GM_\odot}{r} - 2c_s^2 \right) = 0, \quad (10)$$

using polytropic equation (1) and mass-flux relation (8), which shows that at the transonic point, $v(r_s) = c_s(r_s)$ where $2c_s^2(r_s) = \frac{GM_\odot}{r_s}$. Define the gravitational escape speed by $u_g^2(r) = \frac{2GM_\odot}{r}$ at radial distance r . It follows that $v(r_s) = \frac{1}{2}u_g(r_s)$, that is, the flow is accelerated to half the gravitational escape speed at the transonic point. Given F_0 and subject to $\gamma \leq 3/2$, the mathematics of the polytropic wind does not allow $E_0 = \frac{1}{2}v_\infty^2$ to be arbitrarily prescribed but is determined by a self-consistent location of the transonic point $r = r_s$.

The γ -polytrope illustrates two general physical points, modeling the heating ($\gamma < 5/3$) and cooling ($\gamma > 5/3$) of an expanding gas. A static unbounded corona may contain any

amount of total mass M_0 if $\gamma > 4/3$ whereas there is a maximum M_{max} to the total mass M_0 if $\gamma < 4/3$, requiring any excess above which to be ejected to infinity. A steady wind is the natural state of a sufficiently heated corona, represented by the necessary condition $\gamma \leq 3/2$, in which case the terminal speed $v_\infty = \sqrt{2E_0}$ and constant mass flux F_0 are related by a compatibility condition for the location of the transonic surface $r = r_s$ between the subsonic and supersonic parts of the wind.

4. Concluding Remarks

The solar wind is of a hydrodynamic nature^{1,2}, a consequence of ubiquitous heating with heat thermally conducted outward to maintain the supersonic expansion of the outer atmospheric to a terminal speed. Coronal heating is of a magnetic nature. Spontaneous current sheets form by virtue of high electrical conductivity that must dissipate at the implied small electrical resistivity by the unlimited growth of the current density in the thin sheets^{4,5}. In a bootstrapping manner, the liberated heat maintains the fully ionized corona at its million-degree temperature. Although the field dominates over the low to middle tenuous corona, the million-degree plasma pressure declines less slowly with height higher up in the corona than the square of its intensity, the magnetic pressure. The latter faster decline with height is due to the solenoidal nature of the field to connect photospheric foot-points of opposite magnetic polarities. Thus, above the temperature maximum, in the mid-corona, the plasma dominates over the field and carries its frozen-in field into the solar wind. Still, the fields are rooted at the photospheric foot-points, so the dynamically dominating solar wind simply combs the magnetic field into alignment with the flow. Each flow-aligned field line has an enormous length connecting along the flow back to its foot-point in the photosphere.

A point to conclude my Lecture, that coronal thermal conduction is a factor of 10^8 weaker across than along magnetic field lines. Thus, coronal densities generally have strong gradients across field lines. This density anisotropy is seen in the eclipse image Fig. 1(a), revealing the macroscopic bipolar lobes of low-coronal field lines, rimming the occulted Sun, and the open field lines ($\neq 1$) along which the solar wind flows, as sketched in Fig. 1(b).

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Figure

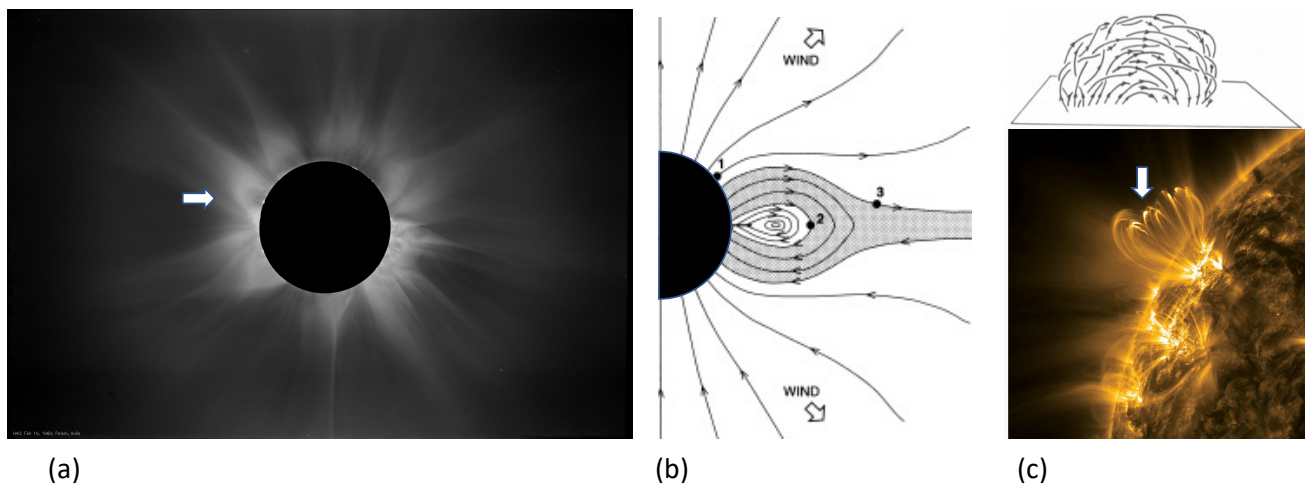


Figure 1. (a): Total eclipse, Feb. 16 1980, India, imaged in solar white-light scattered by coronal electrons; credit, NCAR/Rhode College. (b): Idealized sketch interpreting the magnetic field lines of one of the Sun-rimmed density structures in Fig. 1(a). (c): Idealized sketch of inverted-U magnetic flux-tube such as revealed by its field-aligned coronal plasmas radiating in extreme ultraviolet light; credit, NASA/SDO.